Longo: Math 20C - Winter 2017
Lecture Notes

Date:

Section:

- [ ] §1.3 (cont.)

Topics Covered:

- [ ] The Triple Product the geometry of the cross product, and equations of planes.

Geometry of the Cross Product: Last time we saw that if $\vec{u}=(u_1,u_2,u_3)$, $\vec{v}=(v_1,v_2,v_3)$, the cross product of $\vec{u}$ and $\vec{v}$ is:

$$\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}.$$

Q! What are the magnitude and direction of $\vec{u}\times\vec{v}$?

Before we get there, let's discuss the "triple product" of vectors: Let $\vec{w}=(w_1,w_2,w_3)$, then

$$(\vec{u}\times\vec{v})\cdot\vec{w}=\left(\begin{vmatrix}u_2 & u_3\\ v_1 & v_3\end{vmatrix}\vec{i}-\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}\vec{j}+\begin{vmatrix}u_1 & u_2\\ v_1 & v_3\end{vmatrix}\vec{k}\right)\cdot\left(w_1\vec{i}+w_2\vec{j}+w_3\vec{k}\right)$$

$$=\begin{vmatrix}u_2 & u_3\\ v_1 & v_3\end{vmatrix}w_1-\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}w_2+\begin{vmatrix}u_1 & u_2\\ v_1 & v_3\end{vmatrix}w_3$$

$$=\boxed{\begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}}$$

A simple calculation tells us:

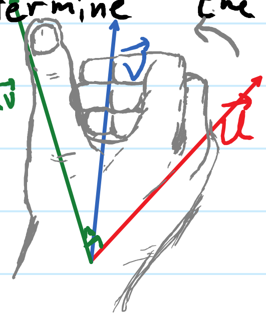
$$(\vec{u}\times\vec{v})\cdot\vec{u}=\begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ u_1 & u_2 & u_3\end{vmatrix}=0.$$ Similarly

$$(\vec{u}\times\vec{v})\cdot\vec{v}=0.$$

This tells us $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

This narrows the direction of $\vec{u}\times\vec{v}$ down to 2 choices. to determine the direction, use the "right hand rule".

This is really the best I can do

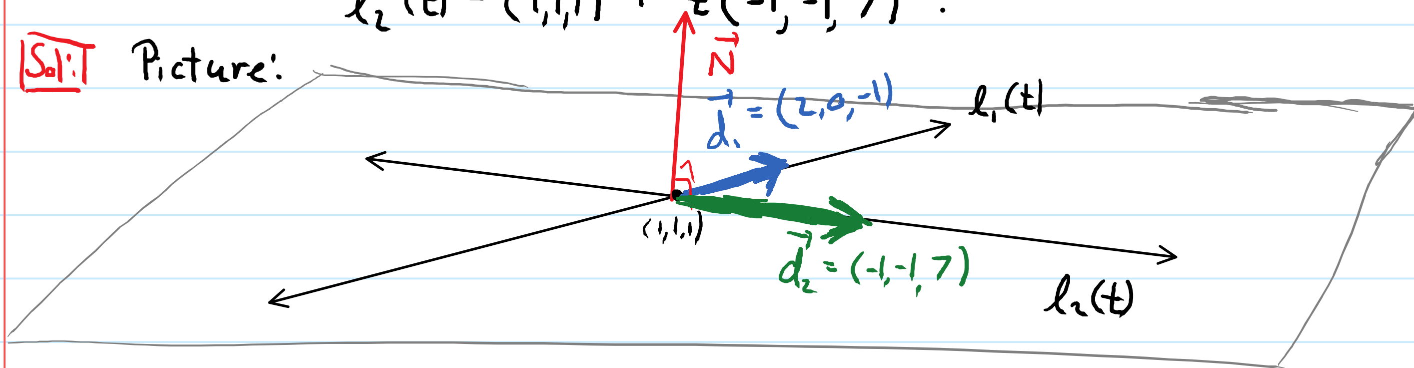


Point your fingers in the direction of $\vec{u}$, then curl them towards $\vec{v}$. Your thumb points at $\vec{u}\times\vec{v}$.

Example: Find a vector $\vec{N}$ that is orthogonal to the plane that contains the lines

$$\vec{\ell}_1(t) = (1,1,1) + t(2,0,-1)$$
$$\vec{\ell}_2(t) = (1,1,1) + t(-1,-1,7).$$

Sol: Picture:



As we see in the picture, $\vec{N}$ is normal to the plane containing $\vec{\ell}_1(t)$, $\vec{\ell}_2(t)$ iff $\vec{N} \perp \vec{d}_1$ and $\vec{N} \perp \vec{d}_2$, where $\vec{d}_1 = (2,0,-1)$ is the direction vector for $\vec{\ell}_1(t)$ and $\vec{d}_2 = (-1,-1,7)$ is the direction vector for $\vec{\ell}_2(t)$.

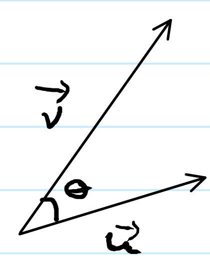
So for $\vec{N}$, we can use

$$\vec{N} = \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ -1 & -1 & 7 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ -1 & 7 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -1 \\ -1 & 7 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} \vec{k}$$

$$\boxed{= (1, -13, -2)}$$

What is $\|\vec{u} \times \vec{v}\|$?

It is not terribly hard to prove:



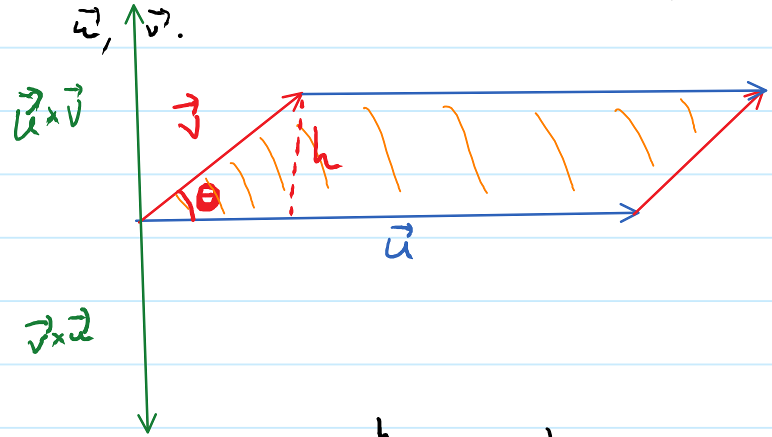
$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$$

where θ is the (smallest) angle between $\vec{u}$, $\vec{v}$.

(See proof on pg. 36 of book. It's not worth doing in class.)

What does this mean?

Let's look at the area of the parallelogram spanned by $\vec{u}, \vec{v}$.



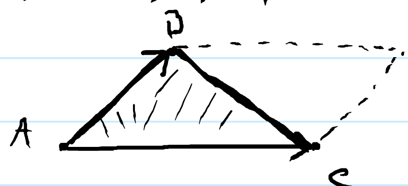
basic trig tells us $\sin(\theta) = \frac{h}{\|\vec{v}\|} \Rightarrow h = \|\vec{v}\| \sin\theta$.

Therefore: $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta =$ Area of the parallelogram.

Example: Find the area of the triangle w/ vertices

$$A = (-1, 3, 4), \quad B = (0, 5, 6), \quad C = (-7, -7, 0).$$

Sol:



The area of ΔABC is $\frac{1}{2}$(parallelogram) spanned by $\vec{AB}, \vec{AC}$. So Area$(\Delta ABC) = \frac{1}{2}\|\vec{AB} \times \vec{AC}\|$.

$$\vec{AB} = \vec{OB} - \vec{OA} = (0-(-1), 5-3, 6-4) = (1, 2, 2)$$
$$\vec{AC} = \vec{OC} - \vec{OA} = (-7-(-1), -7-3, 0-4) = (-6, -10, -4).$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ -6 & -10 & -4 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ -10 & -4 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 2 \\ -6 & -4 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ -6 & -10 \end{vmatrix} \vec{k}$$
$$= (12, -8, 2)$$

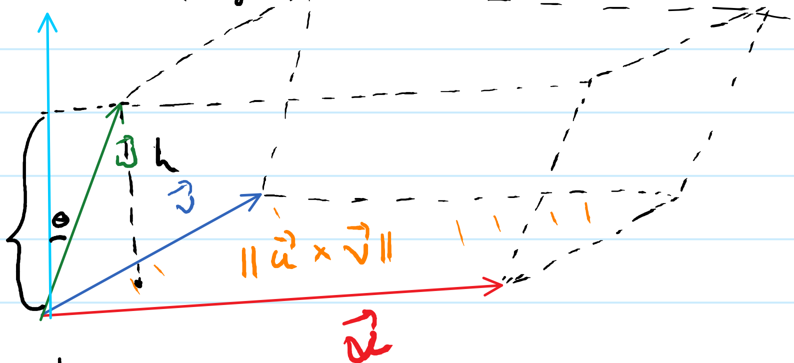
So
$$\frac{1}{2}\|\vec{AB} \times \vec{AC}\| = \frac{1}{2}\left(\sqrt{12^2 + (-8)^2 + 2^2}\right)$$
$$= \frac{1}{2}\sqrt{144 + 64 + 4}$$
$$= \boxed{\frac{1}{2}\sqrt{212}}$$

Geometry of the triple product: $(\vec{u}\times\vec{v})\cdot\vec{\omega}$.

We know: $(\vec{u}\times\vec{v})\cdot\vec{\omega} = \|\vec{u}\times\vec{v}\|\,\|\vec{\omega}\|\cos\theta$ where θ is the angle between $\vec{u}\times\vec{v}$ and $\vec{\omega}$. On the other hand, $\|\vec{u}\times\vec{v}\|$ is the area of the parallelogram spanned by $\vec{u}$, $\vec{v}$, and $\|\vec{\omega}\|\cos\theta = \|\text{proj}_{\vec{u}\times\vec{v}}\vec{\omega}\|$ (see notes from 1-18).

picture:

$h = \|\vec{\omega}\|\cos\theta$



So $\left|(\vec{u}\times\vec{v})\cdot\vec{\omega}\right|$ = volume of the "parallelopiped" spanned by $\vec{u},\vec{v},\vec{\omega}$.

Example: Find the volume of the parallelopiped spanned by $\vec{u}\,(1,1,1)$, $\vec{v}=(-2,0,12)$, $\vec{\omega}=(8,0,0)$

Sol'n: Volume $= \left|(\vec{u}\times\vec{v})\cdot\vec{\omega}\right| = \left|\begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 12 \\ 8 & 0 & 0 \end{vmatrix}\right|$

$= \left|(8)(12)\right| = \boxed{96}$

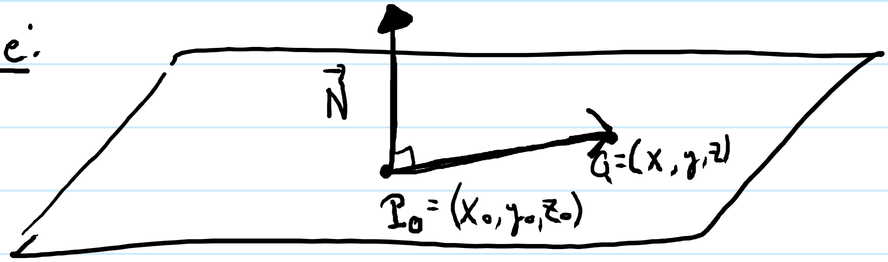
Equations of planes:

To find the eqn of a plane, we need:

1. A point, $P_0 = (x_0, y_0, z_0)$, on the plane.
2. A vector, $\vec{N} = (a, b, c)$, that is normal to the plane. } $\vec{N}$ is called the "normal vector"

Compare this to what we need for the eqn of a line! We can think of $\vec{N}$ as a vector that orients the plane (direction).

Picture:



If Q is any point on the plane, then $\overrightarrow{P_0Q}$ lies in the plane. Since $\vec{N}$ is $\perp$ to the plane, $\vec{N} \perp \overrightarrow{P_0Q}$.
This happens iff $\vec{N} \cdot \overrightarrow{P_0Q} = 0$

$$\Rightarrow \quad \vec{N} \cdot (\overrightarrow{OQ} - \overrightarrow{OP_0}) = 0$$
$$\Rightarrow \quad (\vec{N} \cdot \overrightarrow{OQ}) - (\vec{N} \cdot \overrightarrow{OP_0}) = 0$$
$$\Rightarrow \quad \vec{N} \cdot \overrightarrow{OQ} = \vec{N} \cdot \overrightarrow{OP_0}.$$
$$\Rightarrow \quad ax + by + cz = ax_0 + by_0 + cz_0$$

If we write it out, we have the plane is the set of points (x,y,z) where

$$\boxed{ax + by + cz = d} \quad \text{where } d = ax_0 + bx_0 + cz_0$$

Example: ① Find an eqn of the plane passing through $(9,-2,5)$ with normal vector $\vec{N} = (-6,-6,1)$.

② Find An equation of the plane containing the lines:

$$\vec{l}_1(t) = (1,1,1) + t(2,0,-1)$$
$$\vec{l}_2(t) = (1,1,1) + t(-1,-1,7).$$

$\boxed{\text{Sol:}}$ ① Here we have $P_0 = (9,-2,5)$ and $\vec{N} = (-6,-6,1)$.
So the eqn is.

$$\begin{aligned} -6x - 6y + 1z &= \overrightarrow{OP_0} \cdot \vec{N} \\ &= (9)(-6) + (-2)(-6) + (5)(1) \\ &= -36 + 12 + 5 \\ &= -19 \end{aligned}$$

$$\Rightarrow \quad \boxed{-6x - 6y + z = -19}$$

② We need a pt. P_0 on the plane and a vector $\vec{N}$ that is normal to the plane.

By inspection, we can see $P_0 = (1,1,1)$ is on the plane since it is on the lines.

Earlier, we used the cross product to find that the vector $\vec{N} = (2,0,1) \times (-1,-1,7) = (1,-13,2)$ is $\perp$ to the plane.

So an eqn for the plane is:

$$1x - 13y + 2z = \vec{N} \cdot \overrightarrow{OP_0}$$
$$= (1)(1) + (-13)(1) + (2)(-1)$$
$$= 1 - 13 - 2$$
$$= -14$$

$$\Rightarrow \quad \boxed{x - 13y + 2z = -14}$$