Longo: Math 20C - Winter 2017
Lecture Notes

Date: January 23, 2017

Section:
- [ ] §2.1

Topics Covered:
- [ ] Multivariable functions
- [ ] Graphs of functions
- [ ] Level sets

§2.1: Functions / Graphs / Level sets:

Functions:

Right now, you should be very comfortable with functions f, whose domain is some subset, $\mathcal{U}$, of $\mathbb{R}$, and whose range is some subset of $\mathbb{R}$. We denote such fcns by

$$\boxed{\begin{aligned} f: \mathcal{U} \subseteq \mathbb{R} &\longrightarrow \mathbb{R}. \\ x &\longmapsto y = f(x) \end{aligned}}$$

In this class, we consider multivariable functions f whose domain, $\mathcal{U}$, is a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ and whose range is a subset of $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$. We will greatly empasize functions from $\mathbb{R}^2$ to $\mathbb{R}$, or from $\mathbb{R}$ to $\mathbb{R}^2$ or $\mathbb{R}^3$.

Example: ① $f(x,y) = x^2 + y^2$, $f: \mathbb{R}^2 \to \mathbb{R}$

② $g(x,y,z) = \sqrt{xyz}\, e^{x+y+z}$, $g: \{(x,y,z) \mid xyz \geq 0\} \subseteq \mathbb{R}^3 \to \mathbb{R}$

③ $\vec{\ell}(t) = (1,2,3) + t(-1,0,5)$, $\vec{\ell}: \mathbb{R} \to \mathbb{R}^3$.

Examples ① and ② are called scalar valued because the output is a scalar. Example ③ is vector valued.

Graphs:

In order to understand the behavior of a function, we usually try to visualize the function in the form of a graph.

Def: The graph of a function $f: \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is the set of points of the form $(x, y, f(x,y))$ where (x,y) is a point in $\mathcal{U}$.

Note: ① The graph of a fcn $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a surface in $\mathbb{R}^3$.

② We can define graphs of functions $g: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where n, m are any positive integers similarly.

As we will see, graphs of multivariable fcns are much harder to visualize and draw than single variable fcns. We now discuss tools to help us graph.

Example: $f(x,y) = x^2 + y^2$.

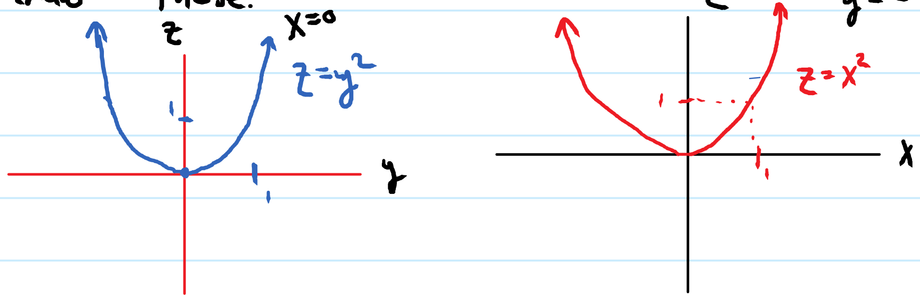
Idea: Instead of trying to graph immediately, we can look at 2-dim. cross sections. We use these to create a wire frame, which will be the skeleton for our graph.

To make cross sections, we fix one of the variables.

Fix $x=0$, then $z = f(0,y) = 0^2 + y^2 = y^2$ is the intersection of the graph with the yz-plane.

Fix $y=0$, then $z = f(x,0) = x^2 + 0^2 = x^2$ is the intersection of the graph with the xz-plane.

These are both just parabolas! We know how to draw those.



Let's try fixing $z=c$ for varying constants c.

If $c=0$, $0=f(x,y)$

$0=x^2+y^2$

the set of (x,y) where $f(x,y)=0$ is $\{(0,0)\}$.

If $c=1$, $1=f(x,y)$

$1=x^2+y^2$

This is the unit circle!

If $c=2$, $2=f(x,y)$

$2=x^2+y^2$

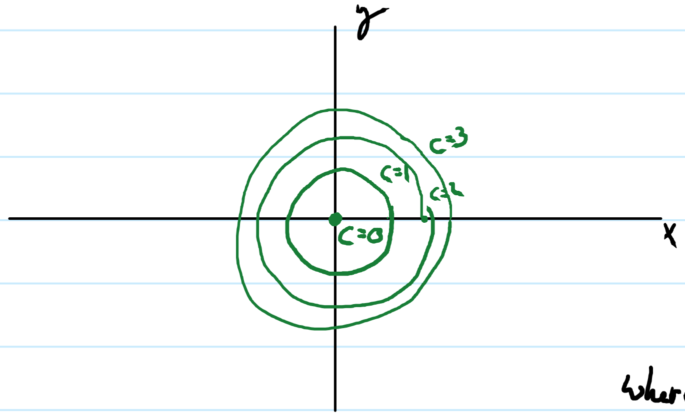
Circle based at the origin with radius $\sqrt{2}$.

If $c=-1$, $-1=f(x,y)$

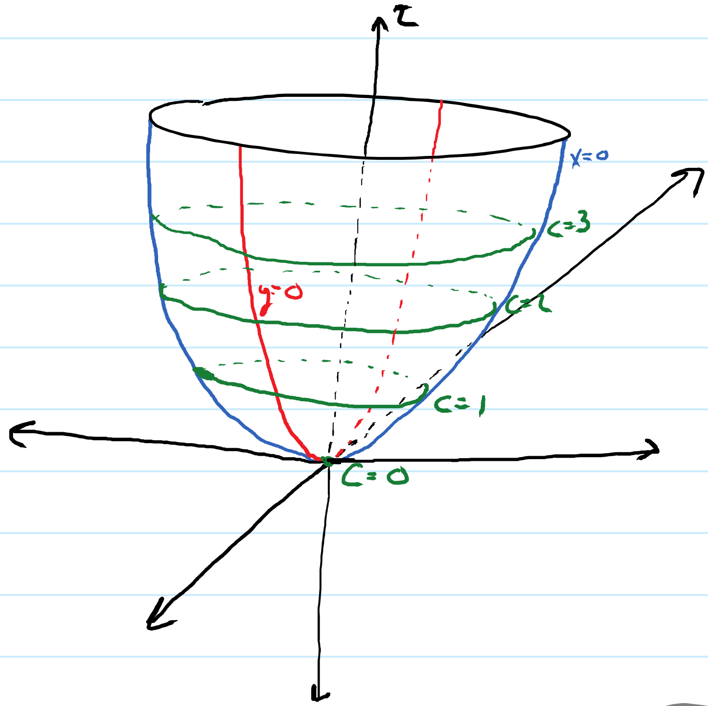
$-1=x^2+y^2$

This is impossible since $x^2+y^2 \geq 0$.

Now, we draw these curves in the xy-plane.



To make the wire frame, imagine the z-axis coming out of the board. Since the circle where $c=1$ is the set of points on the graph where $z=1$, imagine you "pull" the $c=1$ circle out of the board by 1 unit. Pull the circle where $c=2$ out of the board by 2 units, and so on. Before we even sketch it, maybe you can visualize a "bowl shape". This graph is called a paraboloid.



paraboloid.

Notice the graph lies entirely above the xy-plane because $x^2+y^2=C$ is impossible if $C<0$.

Def: The set of points, (x,y), where $f(x,y)=c$ is called the level set or level curve of heigt c (or value c).

Example: Draw level curves of height $-2,-1,0,1,2$ for $f(x,y)=x^2-y^2$. Use them to graph f.

Sol: $c=0$:

$$0=f(x,y)$$
$$0=x^2-y^2$$
$$y^2=x^2$$
$$y=\pm x$$

$\longleftrightarrow$ intersection of two lines through the origin.

$C=1$

$$1=f(x,y)$$
$$1=x^2-y^2$$

$\longleftrightarrow$ hyperbola "opening horizontally"

$C=2$: $2=x^2-y^2$ $\longleftrightarrow$ hyperbola "opening horizontally"

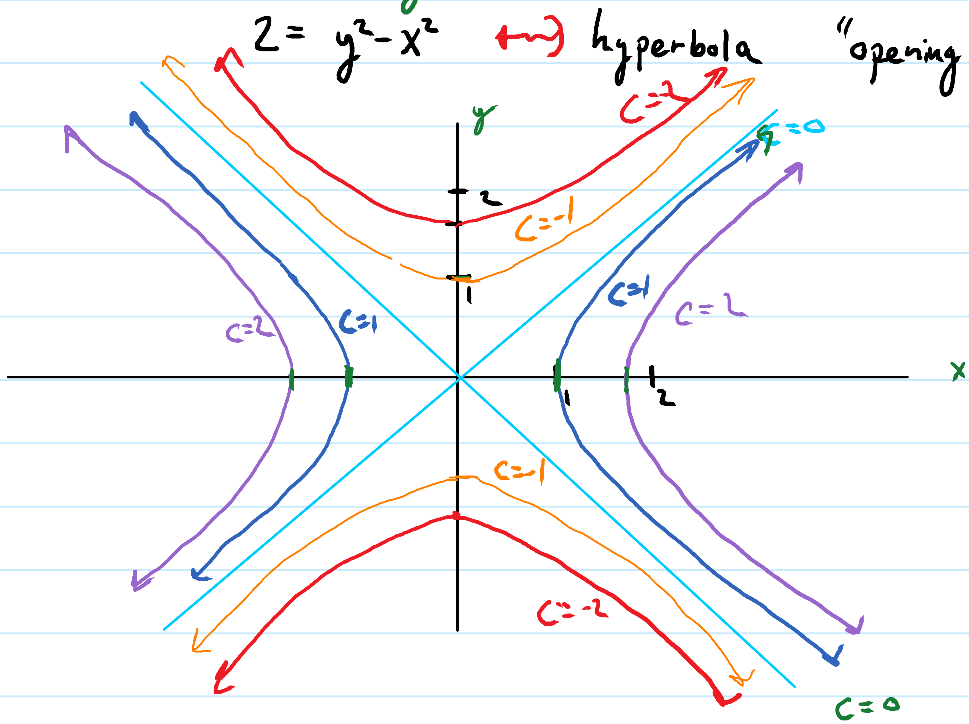
$C=-1$

$$-1=f(x,y)$$
$$-1=x^2-y^2$$
$$1=y^2-x^2$$

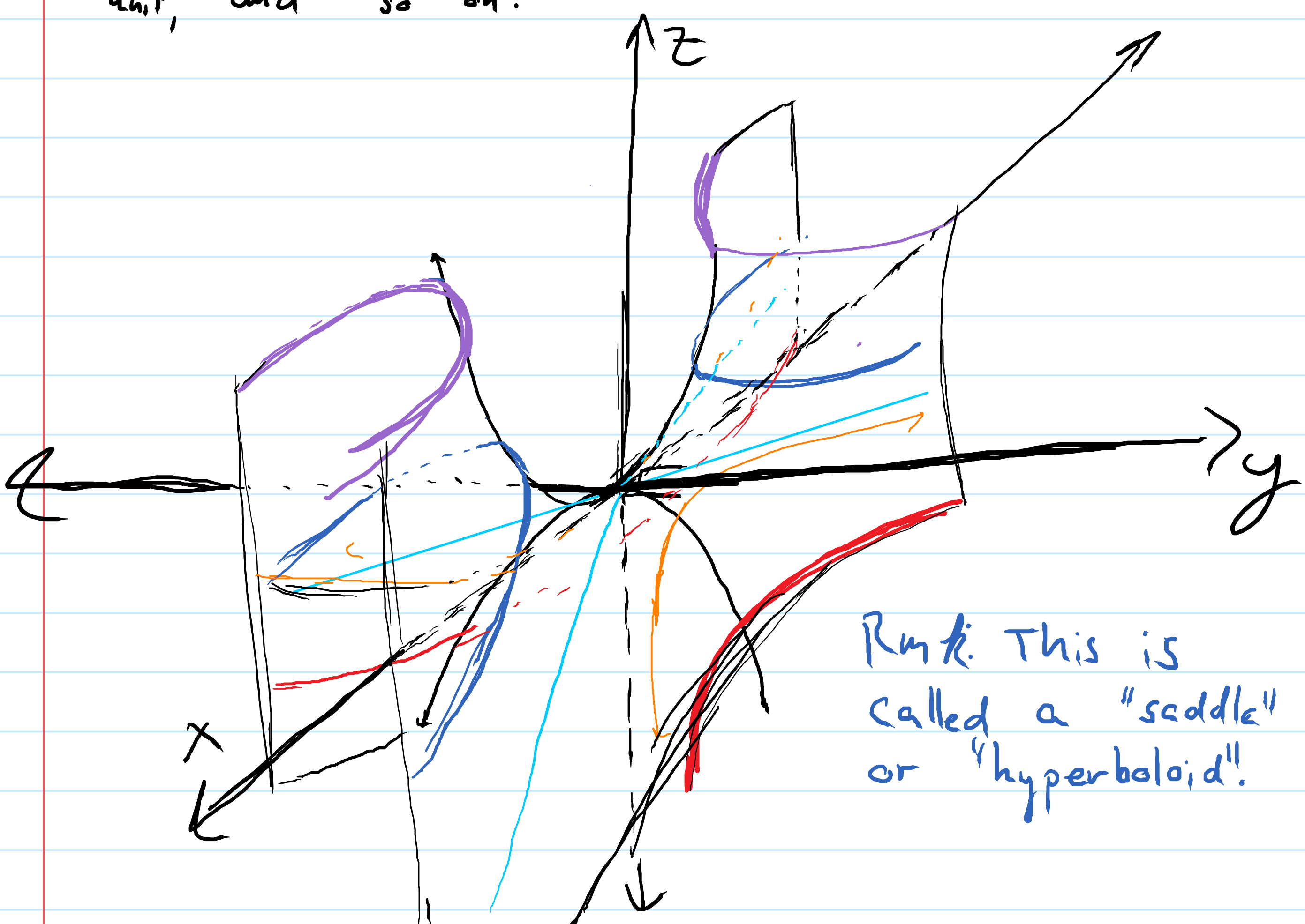
$\longleftrightarrow$ hyperbola "opening vertically"

$c = -2$: $\quad -2 = x^2 - y^2$

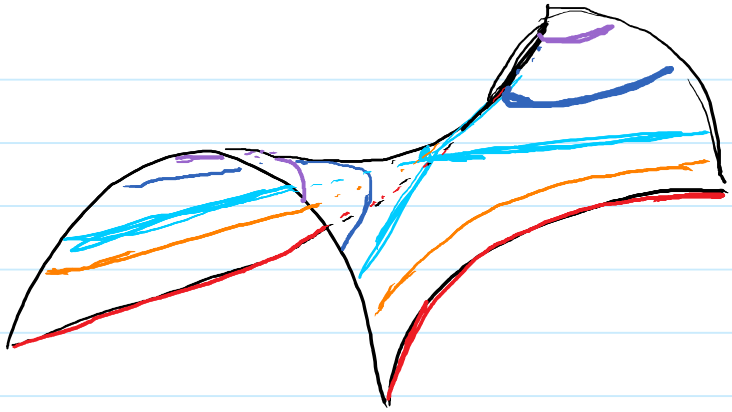
$2 = y^2 - x^2 \iff$ hyperbola "opening vertically"



Then we pull the curve $c=1$ out of the board 1 unit, push the curve $c=-1$ into the board 1 unit, and so on.



Rmk. This is called a "saddle" or "hyperboloid".



More rough drawing.