Longo: Math 20C - Winter 2017
Lecture Notes

Date: January 27, 2017

Section:

- [ ] §2.3 (part I)

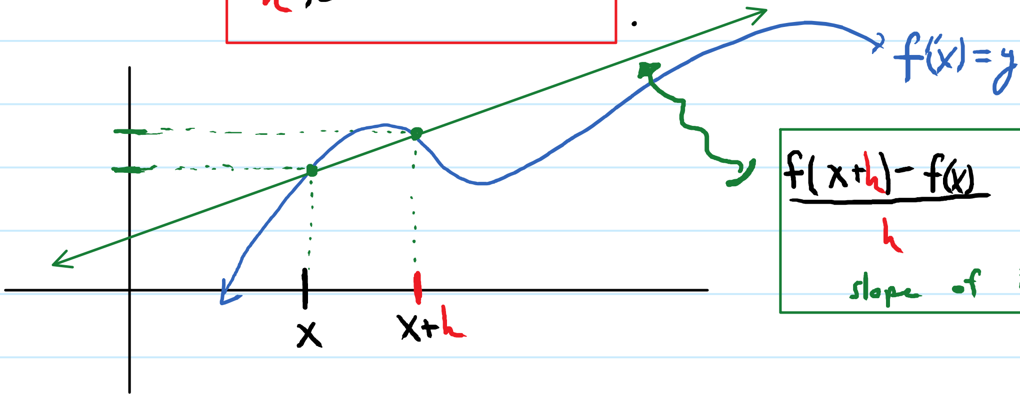
Topics Covered:

- [ ] Partial Derivatives
- [ ] Equation of tangent plane

§2.3 Part 1: Partial derivatives:

Review: If $f: U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a single variable fcn, we define the derivative of f (if it exists) by

$$\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$



$$\boxed{\frac{f(x+h) - f(x)}{h} \text{ is the slope of this line}}$$

$f'(x)$ tells you the instantaneous rate of change of f with respect to a (infinitesimally small) change. I.e., $f'(x)$ is the slope of the tangent line of f at x. To define $f'(x)$, take slope of the secant lines, and let the change in x, h, go to 0.

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Problem: In single variable calculus, we measure change in f as the input changes. However, there is only one direction that the input can change (left or right). Now, if we have a fcn $f: \mathbb{R}^2 \to \mathbb{R}$, and a point $\vec{x}_0 = (a, b)$, it is not clear what we mean by "the derivative of f" since there are infinitely many directions that we can deviate from the point $\vec{x}_0$, and each choice of direction could very well give us a different change in f.

A modest starting point for us will be:

Partial Derivatives: For the sake of visualizing the geometry, assume, for now, $f: \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$.

Idea: Since f has to input values: x, y, we will start by measuring the change in f w.r.t. a small change in the x or y-variable, while the other variable remains fixed.

Def: Let $f: \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$. The x partial derivative / y partial derivative of f (or partial of f w.r.t. x / y) is defined by:

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \boxed{\lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}}\ , \quad \boxed{\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}}$$

More generally, if
$$f: \mathbb{R}^n \to \mathbb{R}$$
$$\vec{x} = (x_1, \dots, x_n) \longmapsto f(x_1, \dots, x_n)$$

We can take partial derivatives w.r.t. any variable

$$\boxed{f_{x_i}(\vec{x}) = \frac{\partial f}{\partial x_i}(\vec{x}) = \lim_{h \to 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h}}$$

To actually calculate partial derivatives (w.r.t. say x) just pretend all the other variables are constant and derive as usual.

Examples: ① $f(x,y) = x^2y - 2xy\ln(y)$.

- $\frac{\partial f}{\partial x} = 2xy - 2y\ln(y)$
- $\frac{\partial f}{\partial y} = x^2 - 2x\ln(y) - 2xy\left(\frac{1}{y}\right)$ ⟵⟶ Product rule!

$= x^2 - 2xy\ln(y) - 2x$

② $g(x,y,z) = \frac{xyz}{x^2+y^2+z^2}$.

- $\frac{\partial g}{\partial z} = \frac{(x^2+y^2+z^2)(xy) - (xyz)(2z)}{(x^2+y^2+z^2)^2}$ ⟵⟶ Quotient Rule.
- $\frac{\partial g}{\partial x} = \frac{(x^2+y^2+z^2)(yz) - (xyz)(2x)}{(x^2+y^2+z^2)^2}$
- $\frac{\partial g}{\partial y} = \frac{(x^2+y^2+z^2)(xz) - (xyz)(2y)}{(x^2+y^2+z^2)^2}$

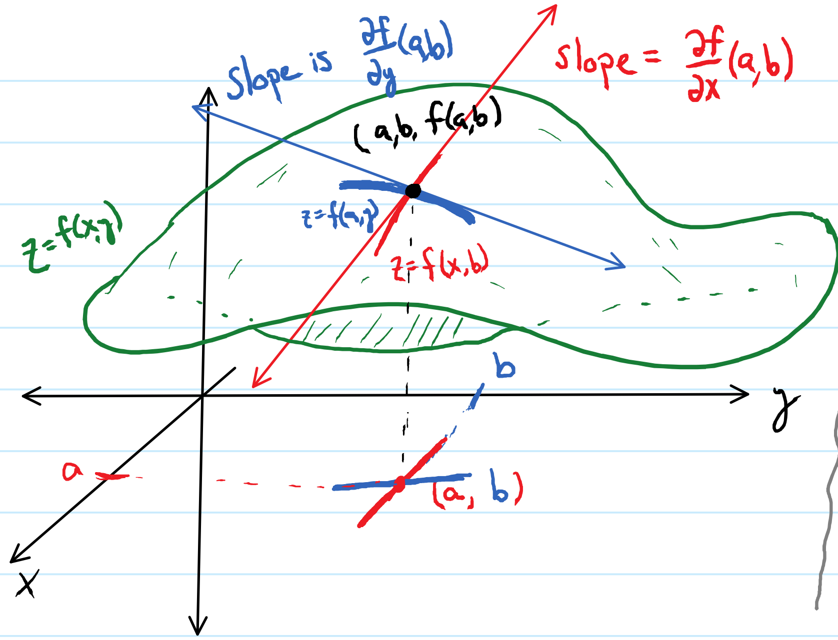
Example: with g as above:

$$\frac{\partial g}{\partial z}(1,1,0) = \frac{(1^2+1^2+0^2)(1\cdot 1) - ((1)(1)(0))(2(0))}{(1^2+1^2+0^2)^2}$$

$$= \frac{2}{4} = \frac{1}{2}$$

$\boxed{Q?}$ What do the partials tell us?

Let $f : \mathbb{R}^2 \to \mathbb{R}$, and fix a point $\vec{x}_0 = (x, b)$ in the domain of f. $\frac{\partial f}{\partial x}(a,b)$ tells us the (instantaneous) rate of change of f as x changes and $y=b$ remains fixed.



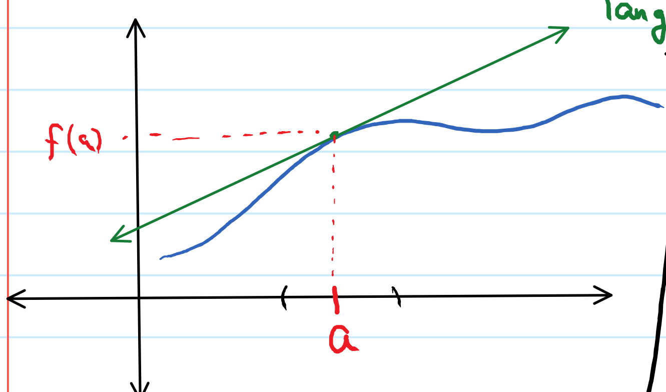
If we look at the points near (a,b) where x varies and $y=b$ is fixed, we have a line segment in the xy-plane $\parallel$ to x-axis. If we plug those points into f, we get a 2D curve sitting on the graph of f. The slope of the tangent line to that curve at (a,b) is the slope of that curve. Similarly, the slope of the blue line is $\frac{\partial f}{\partial y}(a,b)$.

Since the graph of $f(x,y)$ is a 2D-shape sitting in $\mathbb{R}^3$, these two (1D) lines don't tell the whole story of how f changes as you deviate from (a,b). So $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ individually "aren't enough".

$\boxed{Q?}$ What should we call the "derivative of f"? What does it mean for a fcn $f:\mathbb{R}^2\to\mathbb{R}$ to be differentiable?

Let's look back at the single variable case:

Tangent Line: $L(x) = f(a) + f'(a)(x-a)$.



A fcn $f:\mathbb{R}\to\mathbb{R}$ is differentiable at a if the following limit exists

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

What does this tell us?

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \Rightarrow \lim_{x\to a} f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$\Rightarrow 0 = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} - f'(a)$$

$$\Rightarrow 0 = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} - \frac{f'(a)(x-a)}{x-a}$$

$$\Rightarrow 0 = \lim_{x\to a} \frac{f(x) - \overbrace{(f(a)+f'(a)(x-a))}^{L(x)}}{x-a}$$

$$\Rightarrow \boxed{0 = \lim_{x\to a} \frac{f(x)-L(x)}{x-a}}$$

In plain English, this says as x gets close to a, $f(x)$ gets very close to $L(x)$, the fcn whose graph is the tangent line of f at a. I.e., the tangent line at a is a good approximation for f.

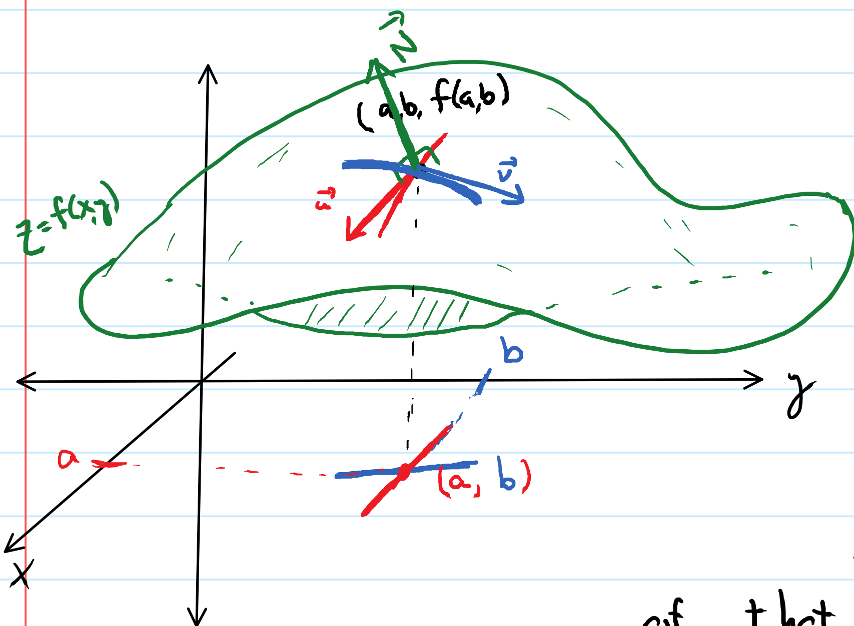
Using this as motivation, we would like to say $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b) if the tangent plane of f at (a,b) is a good approximation for $f(x,y)$. Warning: It is possible for a fcn to have all partial derivatives, and still not be differentiable. Existence of partial derivatives is not enough.

Computing the equation of the Tangent plane of $f(x,y)$ at (a,b).

As we have seen, in order to get the eqn of the tangent plane, we need: ① A point on the plane: P_0
② A vector $\perp$ to the plane: $\vec{N}$

Since the tangent plane touches the graph of f at $(a, b, f(a,b))$. We let $P_0 = (a, b, f(a,b))$.

To find $\vec{N}$, we will find two vectors in the plane, and cross them (just like before). We need two vectors tangent to the graph.



Let

$\vec{u} = (1, 0, \frac{\partial f}{\partial x}(a,b))$

$\vec{v} = (0, 1, \frac{\partial f}{\partial y}(a,b))$.

These vectors work because $\frac{\partial f}{\partial x}$ is the slope of the tan. line of the red curve on the plane, so $\vec{u}$ is the direction vector of that tangent line. Similarly, $\vec{v}$ is the direction vector of the line tangent to the blue curve $f(a,y)$. So we take

$\vec{N} = \vec{u} \times \vec{v}$

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x}(a,b) \\ 0 & 1 & \frac{\partial f}{\partial y}(a,b) \end{vmatrix} = \vec{i} \begin{vmatrix} 0 & \frac{\partial f}{\partial x}(a,b) \\ 1 & \frac{\partial f}{\partial y}(a,b) \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & \frac{\partial f}{\partial x}(a,b) \\ 0 & \frac{\partial f}{\partial y}(a,b) \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= -\frac{\partial f}{\partial x}(a,b)\vec{i} - \frac{\partial f}{\partial y}(a,b)\vec{j} + \vec{k}$$

$$= \left(-\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b), 1\right)$$

So the plane satisfies the eqn:

$$-\frac{\partial f}{\partial x}(a,b)\, x - \frac{\partial f}{\partial y}(a,b)\, y + z = \vec{N} \cdot \vec{OP_0}$$

$$= -\frac{\partial f}{\partial x}(a,b) \cdot a - \frac{\partial f}{\partial y}(a,b) \cdot b + f(a,b)$$

$$\Rightarrow \boxed{z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).}$$

(To be continued...)