Longo: Math 20C - Winter 2017
Lecture Notes

Date: January 26, 2017

Section:
- [ ] §2.3 (part 2)

Topics Covered:
- [ ] Linear approximation
- [ ] The meaning of differentiability, and a criterion for differentiability
- [ ] The total derivative

From Last time: If $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is "nice" (i.e., the graph is smooth) then the equation $z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$ should describe the plane tangent to the graph at (a,b).

Example: Find the eqn of the plane tangent to the graph of $f(x,y) = x^2 - 2xy + y^2$ at the point where $x=1$, $y=2$

Sol:
- $\frac{\partial f}{\partial x} = 2x - 2y \Rightarrow \frac{\partial f}{\partial x}(1,2) = 2 - 2\cdot 2 = 2 - 4 = -2$
- $\frac{\partial f}{\partial y} = -2x + 2y \Rightarrow \frac{\partial f}{\partial y}(1,2) = -2(1) + 2(2) = -2 + 4 = 2$
- $f(1,2) = 1^2 - 2(1)(2) + (2)^2 = 1 - 4 + 4 = 1$

$$\Rightarrow \boxed{z = 1 - 2(x-1) + 2(y-2)}$$

Differentiability ($f: \mathbb{R}^2 \to \mathbb{R}$ case) & linear approximation:

Let $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$, (a,b) be a point in the domain of f. Assume $\frac{\partial f}{\partial x}(a,b)$, $\frac{\partial f}{\partial y}(a,b)$ exist.

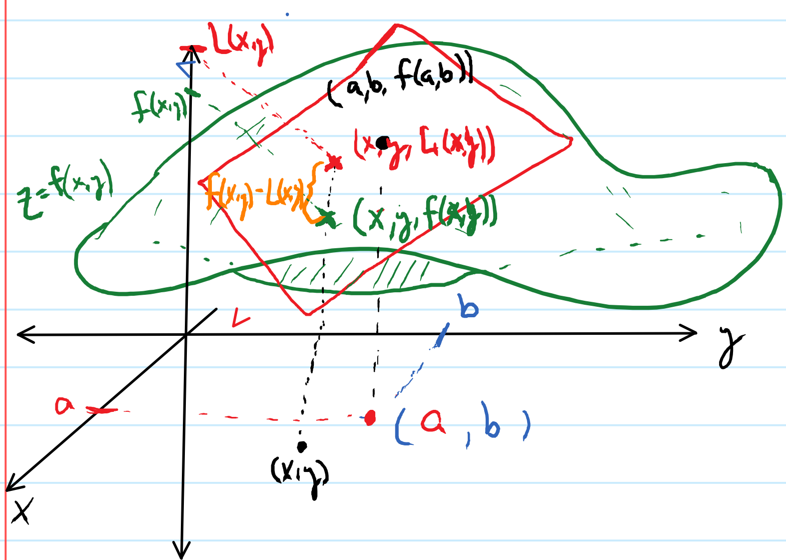
Let $L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$.

$L(x,y)$ is the fcn whose graph is the plane tangent to the graph of f at (a,b).

We saw last time that a fcn of one variable, $g(x)$, differentiable at a point $x=a$ iff the tangent line is a "good approximation" for g. We say $f(x,y)$ is differentiable at (a,b) if $f(x,y) \approx L(x,y)$ if (x,y) is close to (a,b). More precisely:

Definition: In the above setting. f is differentiable at (a,b) if

$$(*) \quad \boxed{\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - L_1(x,y)|}{\|(x,y)-(a,b)\|} = 0}$$



Rmk: We will call $L(x,y)$ the linear approximation of f near (a,b)

Problem: It is hard to check property $(*)$ directly.
Luckily, we have:

Thm: If $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist and are continuous "near" (a,b), then f is differentiable at (a,b).

Rmk: A fcn that has continuous partials is called C^1.
This thm says $C^1 \Rightarrow$ differentiable.

Example: Estimate $(0.99)^2 - 2(0.99)(2.01) + (2.01)^2$ using linear approximation.

Sol: Let $f(x,y) = x^2 - 2xy + y^2$. We want to estimate $f(0.99, 2.01)$.
To do this we use Linear approximation near $(1,2)$.

We saw earlier: $\frac{\partial f}{\partial x} = 2x - 2y$, $\frac{\partial f}{\partial y} = -2x + 2y$, which are continuous everywhere. By the Thm, f is diff'ble, so $f(x,y) \approx L(x,y) = f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2)$

near $(1,2)$. We calculated earlier:

$$L(x,y) = 1 - 2(x-1) + 2(y-2).$$

Since $(0.99, 2.01)$ is close to $(1,2)$

$$f(0.99, 2.01) \approx L(0.99, 2.01) = 1 - 2(-0.01) + 2(0.01)$$
$$= 1 + 0.02 + 0.02$$
$$= \boxed{1.04}$$

Rmk: Existance of partials is not good enough for differentiability.
for ex. $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ has partials everywhere,

but is not diff'ble at $(0,0)$ (See book pg. 114 for picture).

Differentiability (general case) and the total derivative (a survey):

Let $f(x,y)$ be diff'ble at (a,b), $L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$ be the linear approximation near (a,b).

Let $\nabla f(a,b) = \left[\frac{\partial f}{\partial x}(a,b) \quad \frac{\partial f}{\partial y}(a,b) \right]$ be the 1×2 matrix

of partial derivatives. Then, (*) $L(x,y) = f(a,b) + \nabla f(a,b) \cdot (x-a, y-b)$

dot product

This should look familiar: If $g(x)$ is a one variable diff'ble fcn, $y = g(a) + g'(a)(x-a)$ describes the tangent line at $x=a$.
In (*), $\nabla f(a,b)$ is playing the same roll as the derivative in the single variable case!

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is called the gradient of f.

Rmk: Intuitively, in order to fully encapsulate how f changes w.r.t. to a deviation from (a,b), we need to know how f changes w.r.t. x and y. So we need a matrix to hold multiple pieces of information.

More generally: Let $f:\mathbb{R}^n \to \mathbb{R}^m$. Then if $\vec{x}=(x_1, x_2, \dots, x_n)$ is an arbitrary point in the domain, we write

$$f(\vec{x}) = \left(f_1(\vec{x}), f_2(\vec{x}), \dots, f_m(\vec{x}) \right)$$

each of these is a function of n-variables

Example: $f:\mathbb{R}^2 \to \mathbb{R}^3$, $\quad f(x,y) = (\underbrace{2x+y}_{f_1}, \underbrace{3xy}_{f_2}, \underbrace{x^2-y^2}_{f_3})$

Assume each of the f_i's has continuous partial derivatives for each input variable. Then the theorem discussed before says f is differentiable. What does this mean?

Let $\vec{a} = (a_1, \dots, a_n)$ be a vector in $\mathbb{R}^n$ (the domain of f), and let

$$[Df](\vec{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{a}) & \frac{\partial f_1}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{a}) \\ \frac{\partial f_2}{\partial x_1}(\vec{a}) & \cdots & & \frac{\partial f_2}{\partial x_n}(\vec{a}) \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{a}) & \cdots & \cdots & \frac{\partial f_m}{\partial x_n}(\vec{a}) \end{bmatrix}$$

Then for all points $\vec{x}$ "close to" $\vec{a}$ (i.e., $\|\vec{x}-\vec{a}\|$ is small),

$$(**) \qquad f(x) \approx f(\vec{a}) + [Df](\vec{a}) \cdot (\vec{x}-\vec{a})$$

Matrix multiplication

In this equation, we view these as column vectors so that matrix multiplication makes sense.

Here, the matrix of partial derivatives $[Df](\vec{a})$ is called the (total) derivative or the differential of f at $\vec{a}$.

In the special case $m=1$, and we have $f: \mathbb{R}^n \to \mathbb{R}$,

$[Df](\vec{x})$ is just the gradient: ∇f.

Rmk: ① We will use the total derivative when we talk about the multivariable chain rule.

② We will only really consider fcns $f: \mathbb{R}^n \to \mathbb{R}^m$ where $n, m \leq 3$.

Example from before:

Let $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (2x+y,\ 3xy,\ x^2-y^2)$.

Then $[Df](x,y) = \begin{pmatrix} 2 & 1 \\ 3y & 3x \\ 2x & -2y \end{pmatrix}$

(**) More precisely:

$$\lim_{\vec{x}\to\vec{a}} \frac{\|f(\vec{x}) - (f(\vec{a}) + [Df](\vec{a})\cdot(\vec{x}-\vec{a}))\|}{\|\vec{x}-\vec{a}\|} = 0$$

Note that: ① the numerator is a vector in $\mathbb{R}^m$, so

"$f(\vec{x}) \approx f(\vec{a}) + [Df](\vec{a})\cdot(\vec{x}-\vec{a})$" means the difference vector has small magnitude.

② This limit criterion is the official condition for a funtion to be differentiable.