MIDTERM 2 SAMPLE PROBLEMS

(1) Find the extrema of the function $f(x, y) = x^3 - y^3$ on the domain $D = \{(x, y)|x^2 + 3y^2 \leq 1\}$. Does f have extrema on the domain $C = \{(x, y)|x^2 - y = 1\}$?

(2) A function $f : \mathbb{R}^2 \to \mathbb{R}$ is called *harmonic* if $f_{xx} + f_{yy} = 0$. Which of the following functions are harmonic?
  (a) $f_1(x, y) = x^2 - y^2$,
  (b) $f_2(x, y) = xy - yx$,
  (c) $f_3(x, y) = e^x \cos(yx)$.
  (d) $f_4(x, y) = \ln(x^2 - xy + y^3)$.

(3) Let $\mathbf{c}(t)$ be a path function. Show that $\mathbf{c}(t)$ is speeding up at time $t = t_0$ if and only if the velocity vector at t_0 makes an acute angle with the acceleration vector $\mathbf{a}(t)$.

(4) Does there exist a function $f(x, y)$ with continuous second partial derivatives such that $\frac{\partial f}{\partial x} = 5x^2 - y$, and $\frac{\partial f}{\partial y} = xy - 2x$?

(5) Find an equation for the tangent plane to the surface $x^3y - yz^2 + z^4 = 12$ at the point $(2, -1, 2)$. What point on the paraboloid $z = x^2 + y^2$ has a tangent plane that is parallel to the plane you found above?

(6) Find all critical points of the function $f(x, y) = x^2 - 3xy + 5x - 2y + 6y^2 + 8$, and classify each critical point as a local maximum, local minimum, or a saddle point.

(7) Let $f : \mathbb{R}^4 \to \mathbb{R}^2$, and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
$$f(x, y, z, w) = (xy, 2ywz)$$
and
$$g(x, y) = (x^2(2y - 1)^3, x^2y^2).$$
Compute $[\mathbf{D}f](1, 2, 2, -1)$, $[\mathbf{D}g](2, -8)$, and $[\mathbf{D}(g \circ f)](1, 2, 2, -1)$. Explain, without using the word "matrix," why it does not make sense to compute $[\mathbf{D}(f \circ g)](2, -8)$

(8) A farmer from Omaha, Nebraska (A.K.A. *The Jewel of the Northwest*) is walking around in his corn field. Suppose his position at time t-minutes is given by the path function $\mathbf{c}(t) = (t \sin(t), t \cos(t))$. Suppose the deliciousness of the farmers corn at the point (x, y) is given by the function $d(x, y) = 2000e^{-(x-2)^2-(y+1)^2}$.
  (a) After 8 minutes, is the corn becoming more delicious or less delicious at the farmer's position?
  (b) If the farmer is standing at the point $(1, 5)$ in which direction should the farmer walk so that the deliciousness of the corn increases most rapidly?
  (c) If the farmer is standing at the point $(1, 5)$, and if he is satisfied with the deliciousness of his corn, in which direction should he walk so that the deliciousness of the corn does not change? Note, there are two correct answers.
  (d) Where is the most delicious corn in the field? (*Optional:*) Explain how you know your answer is a global maximum.

(9) A bee is flying along the path $\mathbf{c}(t) = (t^2, \cos(t) + t^2, \sin(t))$, where t is measured in seconds. Where is the bee after $\pi/2$ seconds? Find the equation of the line that is tangent to the bee's path at $\pi/2$ seconds. Use linear approximation to estimate the position of the bee after $\pi/2 + 0.1$ seconds.

(10) Find a parametrization of the circle of radius 2 centered at the point $(1, 5)$ that has constant speed 11.

(11) Let $h(x,y,z) = 4xy^2 + \sqrt{xyz} - yz$. Let $P_0 = (1,2,1)$ What is the rate of change of h as you travel from P_0 in the direction parallel to the vector $(3,4,5)$.

(12) Let

$$\mathrm{rec}(r,\theta) = (r\cos\theta, r\sin\theta), \ \mathrm{pol}(x,y) = (\sqrt{x^2+y^2}, \tan^{-1}(y/x))$$

be the functions that converts points in $\mathbb{R}^2$ from polar coordinates to rectangular coordinates and vice versa. Let $P_0 = (2,-15)$ (in rectangular coordinates).

(a) At the point P_0, how will the radius, r, change as y increases?

(b) How does r change as x increases?

(c) What is the rate of change of r as you go from P_0 to the point $(3,-17)$?

(d) How does y change as θ increases? (Hint: convert P_0 to polar coordinates first).

(13) Find all values of k that will force the function $f(x,y) = (k-1)x^2 + kxy + y^2$ has a local minimum at $(0,0)$.

(14) Find the maximum of the function

$$G(x,y) = \frac{1}{4}x^4 - \frac{5}{3}x^3 + y^3 + 3x^2 - \frac{3}{2}y^2 + 20$$

on the region in the xy-plane that is bound by the lines $y=1$, $y=0$, $x=0$, $x=1$.