# Sample Final 1 Solutions

B. M. Longo

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1. In order to find the equation of any plane, we need a point, $P_0 = (x_0, y_0, z_0)$, on the plane, and a vector $\overrightarrow{N} = (a, b, c)$, that is *normal* to the plane. If we have these two pieces of information, then the equation of the plane is:
$$ax + by + cz = d$$
where $d = \overrightarrow{N} \cdot \overrightarrow{OP_0}$.

   For this problem, we can take $P = (1, 1, 1)$ to be the point on the plane since $P$ is on the first line, and therefore it is on the plane. To find a vector that is orthogonal to the plane, we will first find two vectors parallel to the plane, and then take their cross product. Notice that the direction vector $\mathbf{d} = (1, 2, 3)$ is parallel to the plane, and the difference vector $\mathbf{u} = (1, 2, 3) - (1, 1, 1) = (0, 1, 2)$ is on the plane since this vector connects a point on line 1 to a point on line 2 (draw a picture to convince yourself why this works). Therefore, we take
$$\mathbf{N} = \mathbf{d} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{vmatrix} = (1, -2, 1).$$
Finally, we get the equation
$$x - 2y + z = (1, -2, 1) \cdot (1, 1, 1) \implies x - 2y + z = 0.$$

2. To find the equation of a line, we need a direction vector $\mathbf{d}$ and a point $P$ on the line. Then the parametrization is given by $\ell(t) = \overrightarrow{OP} + t\mathbf{d}$. For this problem, we can take $P$ to be **any** point $(x, y, z)$ that satisfies both equations simultaneously. For example, we can take $P = (2, 0, 1)$.

I found this point by arbitrarily setting the $y$ coordinate to be 0, and then solving the result system of linear equations.

Next, to find $\mathbf{d}$, we note that since $\mathbf{d}$ parametrizes the line of intersection between these two planes, $\mathbf{d}$ must be parallel to both planes, and therefore $\mathbf{d}$ must be orthogonal to both normal vectors $(2,-1,1)$ and $(1,1,-1)$. Therefore, we can take

$$\mathbf{d} = (2,-1,1) \times (1,1,-1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = (0,3,3).$$

Finally, we get a parametrization for the line:

$$\ell(t) = (2,0,1) + t(0,3,3).$$

3. In the context of this problem, the **composition function** $(f \circ \mathbf{c})(t)$ is a function that tells us the light intensity at the eel's position at $t$ seconds. We are then being asked to find $(f \circ \mathbf{c})'(5)$. By the chain rule,

$$(f \circ \mathbf{c})'(5) = (\nabla f)(\mathbf{c}(5)) \cdot \mathbf{c}'(5).$$

A quick calculation shows

$$\nabla f = \left(-\frac{1}{2}e^{-x/2}, -\frac{1}{3}e^{-y/3}, -\frac{1}{4}e^{-z/4}\right)$$

and therefore

$$(\nabla f)(\mathbf{c}(5)) = (\nabla f)(1,0,2) = \left(-\frac{1}{2}e^{-1/2}, -\frac{1}{3}, -\frac{1}{4}e^{-1/2}\right).$$

Finally,

$$(f \circ \mathbf{c})'(5) = \left(-\frac{1}{2}e^{-1/2}, -\frac{1}{3}, -\frac{1}{4}e^{-1/2}\right) \cdot (3, 9\pi, 2) = -2e^{-1/2} - 3\pi.$$

4. (a) Let $\mathbf{c}(t) = (5\cos(2t), t, 5\sin(2t))$ be the position of the particle. Then the velocity vector is

$$\mathbf{v}(t) = \mathbf{c}'(t) = (-10\sin(2t), 1, 10\cos(2t)).$$

Then the speed of the particle at time $t$ is

$$\|\mathbf{v}(t)\| = \sqrt{100\sin^2(2t) + 1 + 100\cos^2(2t)} = \sqrt{101}.$$

Therefore, the total distance traveled from $t = 0$ to $t = 2\pi$ is

$$\int_0^{2\pi} \sqrt{101}dt = 2\pi\sqrt{101}.$$

(b) The displacement between the position at $t = 0$ and $t = 2\pi$ is equal to

$$\mathbf{c}(2\pi) - \mathbf{c}(0) = (5, 2\pi, 0) - (5, 0, 0) = (0, 2\pi, 0).$$

5. Use the formula

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

where $\theta$ is the (smallest) angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. You should get

$$\angle BAC = \cos^{-1}\left(\frac{4}{\sqrt{14}\sqrt{19}}\right),$$
$$\angle ACB = \cos^{-1}\left(\frac{-4}{5\sqrt{19}}\right),$$
$$\angle CBA = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right)$$

6. (a) $g_x(-3,4,5) = -2$, $g_y(-3,4,5) = 1$, $g_z(-3,4,5) = 2$.

(b) The maximum rate of change is $\|\nabla g(-3,4,5)\| = \sqrt{4+1+4} = 3$.

(c) We are being asked to find the directional derivative of $g$ at $(-3,4,5)$ in the direction of the difference vector $(-1,8,1) - (-3,4,5) = (2,4,-4)$. We must normalize this vector to get the vector

$$\mathbf{u} = \frac{1}{\sqrt{4+16+16}}(2,4,-4) = \frac{1}{6}(2,4,-4) = \frac{1}{3}(1,2,-2).$$

Finally, we just have to take the dot product with the gradient

$$[D_{\mathbf{u}}f](-3,4,5) = (\nabla g)(-3,4,5)\cdot\mathbf{u} = (-2,1,2)\cdot\frac{1}{3}(1,2,-2) = -\frac{4}{3}.$$

7. This integral is evaluated over the region in the $xy$-plane where $0 \leq y \leq 2$, $y/2 \leq x \leq 1$. If you fix an $x$-value, and find the bounds of $y$ in terms of $x$, we see that this region can also be described as $0 \leq x \leq 1$, $0 \leq y \leq 2x$. If we switch the order of integration, we get

$$\begin{aligned} \int_0^1 \int_0^{2x} y e^{x^3} dy dx &= \int_0^1 (\tfrac{1}{2} y^2 e^{x^3} |_{y=0}^{y=2x}) dx \\ &= \int_0^1 2x^2 e^{x^3} dx \\ &= \int_0^1 \tfrac{2}{3} e^u du \\ &= \tfrac{2}{3}(e-1). \end{aligned}$$

8. **Clarification:** The girth of the bag is the perimeter of the side of the bag given in the picture.

   Let $x$, $y$, and $z$ be the length, width, and height of the bag respectively. Since the girth plus the length has to be at most 90 inches, we get the constraint function

$$x + 2y + 2z = 90.$$

   Notice that I have $x + 2y + 2z = 90$ and not $x + 2y + 2z/le90$ because obviously we will get the maximal volume when we have equality. Since the volume of the bag is $V(x, y, z) = xyz$, we must maximize the function $V(x, y, z)$ subject to the constraint $x + 2y + 2z = 90$ using the method of Lagrange multipliers for example. (Note: the constraint is **not bounded**. The students in this class were told that they did not have to prove that their answer was indeed a maximum.) Let $g(x, y, z) = x + 2y + 2z$. Then the gradient of $g$ is never zero, and therefore we have to solve the Lagrange equation

$$\nabla V = \lambda \nabla g.$$

   A simple calculation shows that this equation is

$$(yz, xz, xy) = \lambda(1, 2, 2).$$

   Thus, we have the system of equations:

$$\begin{array}{lrl} \textbf{I:} & yz & = \lambda \\ \textbf{II:} & xz & = 2\lambda \\ \textbf{III:} & xy & = 2\lambda \\ \textbf{IV:} & x + 2y + 2z & = 90. \end{array}$$

If we solve for lambda in each of these equations, we get $\lambda = yz = xz/2 = xy/2 \implies 2yz = xz = yx$. These equations tell us that $x = 2y$ and $z = y$ (Note: none of the variables can be 0 in the context of this problem). Substituting back into equation **IV:**, we get $2y + 2y + 2y = 90 \implies y = 15 \implies z = 15$ and $x = 30$. Therefore, we get one critical point: $(20, 15, 15)$. Though it is a bit tough to prove, it should not be hard to convince yourself that these dimensions must give you a maximum volume.