Sample Final 2 Solutions

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1. (a) In order to find the equation of any plane, we need a point, $P_0 = (x_0, y_0, z_0)$, on the plane, and a vector $\overrightarrow{N} = (a, b, c)$, that is *normal* to the plane. If we have these two pieces of information, then the equation of the plane is:

$$ax + by + cz = d$$

where $d = \overrightarrow{N} \cdot \overrightarrow{OP_0}$.

For this problem, we can take $P = (1, 1, 1)$ to be the point on the plane. To find a vector that is orthogonal to the plane containing the triangle, we can use $\overrightarrow{N} = \overrightarrow{PQ} \times \overrightarrow{PR}$. $\overrightarrow{N}$ is normal the plane because it is orthogonal to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, which are both vectors in the plane. We calculate:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (1, 2, 3) - (1, 1, 1) = (0, 1, 2),$$
$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (0, 1, 1) - (1, 1, 1) = (-1, 0, 0).$$

Then

$$\overrightarrow{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ -1 & 0 & 0 \end{vmatrix} = (0, -2, 1).$$

Finally, the equation of the plane is

$$0x - 2y + 1z = (0, -2, 1) \cdot (1, 1, 1) \implies -2y + z = -1.$$

(b) Recall that for any pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, $\|\mathbf{u} \times \mathbf{v}\|$ is the area of the parallelogram that $\mathbf{u}$ and $\mathbf{v}$ span. Since $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are two sides of the triangle, the area of the triangle is given by

$$\frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2}\|(0, -2, 1)\| = \frac{\sqrt{5}}{2}.$$

2. (a) Since $\mathbf{a}(t) = \mathbf{v}'(t)$, we have

$$\mathbf{a}(t) = (2\cos(t^2) - 4t^2\sin(t^2), -2\sin(t^2) - 4t^2\cos(t^2), 2t).$$

(b) Since $\mathbf{r}'(t) = \mathbf{v}(t)$, we can find $\mathbf{r}(t)$ by first integrating each component with respect to t (this determines $\mathbf{r}(t)$ up to a constant vector $\mathbf{u}$. Then we use the initial condition $\mathbf{r}(0) = (1, 0, 1)$ to solve for the constant. Using the substitution method of integration, we see that

$$\mathbf{r}(t) = (\sin(t^2), \cos(t^2), \frac{1}{3}t^3 - t) + \mathbf{u}$$

where $\mathbf{u}$ is a constant vector. By plugging in $t = 0$, we get

$$(1, 0, 1) = \mathbf{r}(0) = (\sin(0), \cos(0), 0) + \mathbf{u} = (0, 1, 0) + \mathbf{u}.$$

Therefore, $\mathbf{u} = (1, -1, 1)$ which gives us

$$\mathbf{r}(t) = (\sin(t^2), \cos(t^2), \frac{1}{3}t^3 - t) + (1, -1, 1).$$

(c) We must use the **arc length formula**:

$$\text{arc length from } t = 1 \text{ to } t = 2 = \int_1^2 \|\mathbf{v}(t)\| dt.$$

Since

$$\begin{aligned} \|\mathbf{v}(t)\| &= \sqrt{(2t\cos(t^2))^2 + (-2t\sin(t^2))^2 + (t^2 - 1)^2} \\ &= \sqrt{4t^2(\cos^2(t^2) + \sin^2(t^2)) + (t^2 - 1)^2} \\ &= \sqrt{4t^2 + t^4 - 2t^2 + 1} \\ &= \sqrt{t^4 + 2t^2 + 1} \\ &= \sqrt{(t^2 + 1)^2} \\ &= t^2 + 1, \end{aligned}$$

the total distance traveled from $t = 1$ to $t = 2$ is

$$\int_1^2 t^2 + 1 dt = (\frac{1}{3}t^3 + t)\|_1^2 = \frac{28}{3}.$$

3. We use **implicit differentiation** and differentiate both sides of the equation with respect to x:

$$\frac{\partial}{\partial x}(xe^z + ze^y) = \frac{\partial}{\partial x}(x+y) \implies$$
$$e^z + xe^z\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x}e^y = 1 \implies$$
$$\frac{\partial z}{\partial x} = \frac{1-e^z}{xe^z + e^y}.$$

Note: This class used a different textbook where implicit differentiation was emphasized more. In particular, using the chain rule, you can show that if $F(x,y,z) = xe^z + ze^y - x - y$, then $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$.

4. Let $F(x,y,z) = \frac{x^2}{4}+y^2+\frac{z^2}{9}$. Then the ellipsoid is just the **level surface** $F(x,y,z) = 1$. Therefore, the tangent plane to the surface at any point (x,y,z) is normal to $\nabla F(x,y,z)$. This is because **the gradient of a function at a point is orthogonal to the level curve at the point**. We are looking for a point (x,y,z) where ∇F is parallel to $(1,1,1)$ (the normal vector for the plane $x+y+z=1$). We have

$$\nabla F = (\frac{x}{2}, 2y, \frac{2z}{9}).$$

If we want this vector to be parallel to $(1,1,1)$, we need $\nabla F = k(1,1,1)$ for some constant k. It is not hard to see that this condition forces the equations

$$\frac{x}{2} = 2y = \frac{2z}{9} \implies x = 4y \text{ and } z = 9y.$$

Since the point (x,y,z) is also on the ellipsoid, we also have the equation

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1.$$

After substituting, we get

$$\frac{(4y)^2}{4} + y^2 + \frac{(9y)^2}{9} = 1 \implies y = \pm\frac{1}{\sqrt{14}}.$$

If we back-substitute into the equations $x = 4y$ and $z = 9y$, we get two points:

$$\frac{1}{\sqrt{14}}(4,1,9) \text{ and } \frac{1}{\sqrt{14}}(-4,-1,-9).$$

5. (a) The formula for the equation for the tangent plane of the graph of f at when $(x, y) = (a, b)$ is given by the equation

$$z = f(a, b)\frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).$$

In this problem $(a, b) = (3, 4)$ (the x and y values). Since

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$$

and

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}},$$

we have

$$f(3, 4) = \sqrt{3^2 + 4^2} = 5, \frac{\partial f}{\partial x}(3, 4) = \frac{3}{5}, \text{ and } \frac{\partial f}{\partial y}(3, 4) = \frac{4}{5}.$$

Using the above formula, we get the equation

$$z = 5 + \frac{3}{5}(x - 3) + \frac{4}{5}(y - 4).$$

(b) Let $L(x, y) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$ be the **linearization** of f at $(3, 4)$. Since $(3.1, 4.2)$ is very close to $(3, 4)$, we have $f(3.1, 4.2) \approx L(3.1, 4.2) = 5+\frac{3}{5}(3.1-1)+\frac{4}{5}(4.2-4) = 5+\frac{3}{5}(0.1)+\frac{4}{5}(0.2) = 5.22$.

6. (a) Lets us first normalize the vector $\mathbf{v}$ to get $\overrightarrow{e_{\mathbf{v}}} = \frac{1}{\sqrt{3}}(-1, 1, 1)$. Then the direction derivative of f at P in the direction of $\mathbf{v}$ is given by the formula:

$$D_{\mathbf{v}}f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \overrightarrow{e_{\mathbf{v}}}.$$

Since $\nabla f(1, 1, 1) = (2xz, -2y, x^2)|_{(x,y,z)=(1,1,1)} = (2, -2, 1)$, we have

$$D_{\mathbf{v}}f(1, 1, 1) = (2, -2, 1) \cdot \frac{1}{\sqrt{3}}(-1, 1, 1) = \frac{-3}{\sqrt{3}}.$$

Since the directional derivative is negative, the function is **decreasing** at P as you move in the $\mathbf{v}$ direction.

(b) The direction of maximal rate of change for the function f at P is $\nabla f(P)$, and the maximum rate of change is $\|\nabla f(P)\|$. In this problem $\nabla f(P) = (2, -2, 1)$. Therefore the maximum rate of increase is $\|(2, -2, 1)\| = \sqrt{9} = 3$.

7. To find the critical points of f, we need to find all points (x, y) where $\nabla f(x, y) = \mathbf{0}$. I.e., we are looking for points where $\frac{\partial f}{\partial x} = 0$ **and** $\frac{\partial f}{\partial y} = 0$ simultaneously. Since $\nabla f = (4x^3 - 4y, -4x + 4y)$, we need to solve the system of equations

$$\begin{aligned} \textbf{I:} \quad & 4x^3 - 4y &= 0 \\ \textbf{II:} \quad & -4x + 4y &= 0. \end{aligned}$$

The second equation implies $x = y$. Plugging this back into equation **I** gives us $4y^3 - 4y = 4(y)(y - 1)(y + 1) = 0$, which implies $y = 0$, 1, or -1. Since $x = y$, we get three critical points: $(0, 0)$, $(1, 1)$ and $(-1, -1)$. To check whether each critical point if a local maximum, local minimum, or a saddle point of f, we use the **second derivative test**. If we calculate second order partial derivatives, we get

$$\frac{\partial^2 f}{\partial x^2} = 12x^2, \ \frac{\partial^2 f}{\partial y^2} = y, \text{ and } \frac{\partial^2 f}{\partial x \partial y} = -4.$$

We can now calculate the **discriminant of** f:

$$D(x, y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \implies$$

$$D(x, y) = (12x^2)(4) - (-4)^2 = 48x^2 - 16.$$

For $(0, 0)$, we have $D(0, 0) < 0$. Therefore f has a saddle point at $(0, 0)$. For $(1, 1)$, $D(1, 1) > 0$ and $\frac{\partial^2 f}{\partial x^2}(1, 1) = 12 > 0$. Therefore f has a local minimum at $(-1, -1)$. For $(-1, -1)$, $D(-1, -1) > 0$ and $\frac{\partial^2 f}{\partial x^2}(-1, -1) = 12 > 0$. Therefore f has a local minimum at $(-1, -1)$.

8. (a) Notice that the constraint function describes an ellipse. Since f is continuous, and the constraint is a closed and bounded subset of $\mathbb{R}^{\not\vDash}$, the **Extreme Value Theorem** guarantees that f attains its maximum and minimum values on the ellipse $4x^2 + 9y^2 = 36$.

(b) Let $g(x,y) = 4x^2 + 9y^2$. Notice that $\nabla g = (8x, 18y) = \mathbf{0}$ only at the point $(0,0)$. Since $(0,0)$ does not satisfy the constraint $4x^2 + 9y^2 = 36$, we can disregard this point. We must therefore move on to solving the **Lagrange equation**

$$\nabla f = \lambda \nabla g$$

where λ is a real number. Since $\nabla f = (2xy, x^2)$, the Lagrange equation becomes

$$(2xy, x^2) = \lambda(4x, 18y).$$

This gives us the system of equations

$$\begin{array}{rrl} \textbf{I:} & 2xy & = 4\lambda x \\ \textbf{II:} & x^2 & = 18\lambda y \\ \textbf{III:} & 4x^2 + 9y^2 & = 36 \end{array}.$$

From equation **I**, we get $xy = 2\lambda x$, which implies either $x = 0$ or $\lambda = y/2$.

If $x = 0$, then equation **III** implies $9y^2 = 36 \implies y = \pm 2$. Therefore, we get two critical points $(0,2)$ and $(0,-2)$. (Notice that if $x = 0$, then equation **II** does not give us any contradictions.)

If $\lambda = y/2$, equation **II** becomes $x^2 = 9y^2$. Then plugging into equation **III**, we get $4(9y^2) + 9y^2 = 36 \implies y^2 = \frac{4}{5} \implies y = \pm\frac{2}{\sqrt{5}}$. Then since $x^2 = 9y^2$, we have four more critical points $(\frac{6}{\sqrt{5}}, \frac{2}{\sqrt{5}})$, $(\frac{6}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$, $(-\frac{6}{\sqrt{5}}, \frac{2}{\sqrt{5}})$, and $(-\frac{6}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$.

Now all you have to do is plug all five critical points back into the function f. You should get that the maximum of f on this constraint is $\frac{72}{5\sqrt{5}} + 1$ and the minimum of f on this constrainst is $1 - \frac{72}{5\sqrt{5}}$.

9. (a) By switching the order of integration, we have

$$\int_0^1 \int_{x^2}^{x} \frac{\sin(\pi y)}{\sqrt{y} - y} dydx = \int_0^1 \int_y^{\sqrt{y}} \frac{\sin(\pi y)}{\sqrt{y} - y} dxdy$$

Before reading the paragraph below, you should just draw a picture of the domain and try to understand how I got the bounds in the integral above.

To see this, notice that in the original integral, the domain is described as $\{(x,y)|0 \le x \le 1, x^2 \le y \le x\}$, which is the region in the xy-plane that is bounded by the curves $y = x$ and $y = x^2$ (draw a picture!). To change the order of integration, we fix a y-value and find the appropriate range for x in the domain. We see that for a given y, $y \ge x \le \sqrt{y}$. Since $0 \le y \le 1$ for any point (x,y) in the domain, we see that the domain can also be described as $\{(x,y)|0 \le y \le 1, y \le x \le \sqrt{y}\}$, which gives us the bounds of integration above.

(b)

$$\begin{aligned}
\textstyle\int_0^1 \int_y^{\sqrt{y}} \frac{\sin(\pi y)}{\sqrt{y}-y} dxdy &= \textstyle\int_0^1 \left(\left(x \frac{\sin(\pi y)}{\sqrt{y}-y} \right) |_{x=y}^{x=\sqrt{y}} \right) dy \\
&= \textstyle\int_0^1 (\sqrt{y} - y) \frac{\sin(\pi y)}{\sqrt{y}-y} dy \\
&= \textstyle\int_0^1 \sin(\pi y) dy \\
&= -\frac{1}{\pi} \cos(\pi y)|_{y=0}^{y=1} \\
&= -\frac{1}{\pi}(\cos(\pi) - \cos(0)) \\
&= -\frac{1}{\pi}(-2) \\
&= \frac{2}{\pi}
\end{aligned}$$

10. **Important note:** This problem would be much easier (almost trivial) if we had talked about double integration using polar coordinates. However, that topic has been moved to Math 20E, and is no longer required in 20C. For this problem, focus mainly on how to set up the double integral.

    That being said, we are looking for the double integral of $f(x,y) = 4 - x^2 - y^2$ over the domain $D = \{(x,y)|x^2 + y^2 \le 1\}$. Note that D can also be described as the set of points (x,y) in the xy- plane where $-1 \le x \le 1$ and $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$. Therefore, we must calculate the double integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 4 - x^2 - y^2 dydx.$$

If we compute the inner integral, we have

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 4-x^2-y^2 dy = ((4-x^2)y - \frac{1}{3}y^3) |_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = 2((4-x^2)\sqrt{1-x^2} - \frac{1}{3}(1-x^2)^{\frac{3}{2}}.$$

Once we plug into the outer integral, we must calculate

$$\int_{-1}^{1} 2((4-x^2)\sqrt{1-x^2} - \frac{1}{3}(1-x^2)^{\frac{3}{2}}dx$$

Which can be done using the substitution $x = \sin(\theta)$. You should get $\frac{7\pi}{2}$.

In case you were curious, if you convert this integral to polar coordinates you get

$$\iint_D 4 - x^2 - y^2 dA = \int_0^{2\pi} \int_0^1 4r - r^3 dr d\theta$$

which is much easier.