Solutions to the first sample midterm 1

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1. In order to find the equation of any plane, we need a point, $P_0 = (x_0, y_0, z_0)$, on the plane, and a vector $\overrightarrow{N} = (a, b, c)$, that is *normal* to the plane. If we have these two pieces of information, then the equation of the plane is:
$$ax + by + cz = d$$
where $d = \overrightarrow{N} \cdot \overrightarrow{OP_0}$.

   In this problem, we can take P_0 to be any point on the line. So let
$$P_0 = \overrightarrow{r_1}(0) = (1, 2, 3)$$
for example.

   To find $\overrightarrow{N}$, we can take two vectors that are parallel to the plane, and cross them. Since the direction vectors, $\overrightarrow{d_1} = (1, -3, -2)$ and $\overrightarrow{d_2} = (2, 2, -1)$, of $\overrightarrow{r_1}(t)$ and $\overrightarrow{r_2}(t)$ respectively are parallel to the plane, we can use:
$$\overrightarrow{N} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & -3 & -2 \\ 2 & 2 & -1 \end{vmatrix}.$$
A calculation similar to question $2(a)$ gives us
$$\overrightarrow{N} = (-7, 3, -8).$$
Since $\overrightarrow{N} \cdot \overrightarrow{OP_0} = (-7, 3, -8) \cdot (1, 2, 3) = -7 + 6 - 24 = -25$, we get an equation of the plane:
$$-7x + 3y - 8z = -25.$$

2. (a)

$$\begin{aligned}
\vec{v} \times \vec{w} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -2 \\ 2 & 3 & -1 \end{vmatrix} \\
&= \vec{i} \begin{vmatrix} 2 & -2 \\ 3 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \\
&= 4\vec{i} - 3\vec{j} - \vec{k} \\
&= (4, -3, -1).
\end{aligned}$$

(b) The area of the parallelogram is exactly $\|\vec{v} \times \vec{w}\| = \|(4, -3, -1)\|$. Therefore the area of the parallelogram is

$$\sqrt{4^2 + (-3)^2 + (-1)^2} = \sqrt{26}.$$

(c) Warning: Since $\vec{w}$ is not necessarily orthogonal to $\vec{v}$, $\frac{1}{\|\vec{w}\|}\vec{w}$ is not a correct response. We first find a vector that is orthogonal to $\vec{v}$ and $\vec{v} \times \vec{w}$ by computing the cross product of these two vectors:

$$\begin{aligned}
\vec{v} \times (\vec{v} \times \vec{w}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -2 \\ 4 & -3 & -1 \end{vmatrix} \\
&= \vec{i} \begin{vmatrix} 2 & -2 \\ -3 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -2 \\ 4 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2 \\ 4 & -3 \end{vmatrix} \\
&= -8\vec{i} - 7\vec{j} - 11\vec{k} \\
&= (-8, -7, -11).
\end{aligned}$$

Now in order to get the **unit vector** we want, we can just normalize the vector we just found. So the vector we want is

$$\frac{1}{\sqrt{(-8)^2 + (-7)^2 + (-11)^2}}(-8, -7, -11) = \frac{1}{\sqrt{234}}(-8, -7, -11).$$

3. We will complete this problem in two different ways. For the first approach, we can use the **Squeeze Theorem**. Notice that since $y^2 \geq 0$, we have

$$\sqrt{x^2 + y^2} \geq \sqrt{x^2} = x.$$

Therefore, we have

$$\frac{2x^2}{\sqrt{x^2+y^2}} \leq \frac{2x^2}{\sqrt{x^2}} = 2x.$$

On the other hand, since $2x^2$ and $\sqrt{x^2+y^2}$ are both greater than or equal to 0, we have

$$\frac{2x^2}{\sqrt{x^2+y^2}} \geq 0$$

for any point $(x, y) \neq (0, 0)$. All together,

$$0 \leq \frac{2x^2}{\sqrt{x^2+y^2}} \leq 2x,$$

and therefore the Squeeze Theorem tells us

$$\lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} \frac{2x^2}{\sqrt{x^2+y^2}} \leq \lim_{(x,y)\to(0,0)} 2x.$$

An easy computation gives us

$$0 \leq \lim_{(x,y)\to(0,0)} \frac{2x^2}{\sqrt{x^2+y^2}} \leq 0$$

and therefore

$$\lim_{(x,y)\to(0,0)} \frac{2x^2}{\sqrt{x^2+y^2}} = 0.$$

The second approach is to use **polar coordinates.**
Recall that in polar coordinates,

$$x = r\cos\theta,\ \ y = r\sin\theta$$

where r is the distance from the origin to (x, y) and θ is the angle from the line connecting the origin to (x, y) and the positive x-axis. In polar coordinates, we have $2x^2 = 2r^2\cos^2\theta$, and

$$\sqrt{x^2+y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = \sqrt{r^2} = r.$$

After we convert to polar coordinates, we have

$$\lim_{(x,y)\to(0,0)} \frac{2x^2}{x^2+y^2} = \lim_{r\to 0} \frac{2r^2\cos^2\theta}{r} = \lim_{r\to 0} 2r\cos^2\theta = 0\cdot\cos^2\theta = 0.$$

Since the answer is independent of the angle θ (i.e., the limit is independent of the path we take to get to $(0,0)$), the limit is 0.

4. (a) Label the four points so that $A = (0,0,0)$, $B = (3,6,-2)$, $C = (5,7,0)$, and $D = (2,1,2)$, and then draw the parallelogram so that the vertex C is opposite the vertex A. We know that this is the case since for example, $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$, and therefore by the parallelogram rule for vector addition, C is the vertex opposite A. If you draw a picture, this is not very hard to visualize.
Recall that for any two vectors $\overrightarrow{u}$, and $\overrightarrow{v}$, we have

$$\overrightarrow{u}\cdot\overrightarrow{v} = \|\overrightarrow{u}\|\|\overrightarrow{v}\|\cos\theta$$

where θ is the smallest angle between the two vectors. Let θ be the angle between $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Using the above identity, we have

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AD}}{\|\overrightarrow{AB}\|\|\overrightarrow{AD}\|}\right).$$

Now, $\overrightarrow{AB} = (3,6,-2)$ and $\overrightarrow{AD} = (2,1,2)$. so we have:

$$\overrightarrow{AB}\cdot\overrightarrow{AD} = 6+6-4 = 8,$$

$$\|\overrightarrow{AB}\| = \sqrt{3^2+6^2+(-2)^2} = \sqrt{49} = 7,$$
$$\|\overrightarrow{AD}\| = \sqrt{2^2+1^2+2^2} = \sqrt{9} = 3.$$

All together,

$$\theta = \cos^{-1}\left(\frac{8}{21}\right).$$

Now to get the other angle, we can notice that the the angle, θ, between $\overrightarrow{AB}$ and $\overrightarrow{AD}$, and the angle, α, between $\overrightarrow{DC}$ and $\overrightarrow{DA}$ add up to $180°$. So to get the other angle, we can just take $\pi - \cos^{-1}(\frac{8}{21})$. If I recall correctly, almost every student that took this exam did the full calculation over again.

(b) There are many ways to do this part of the question. Before you start, you should draw a picture so that it is easier to understand the geometry. Let $\mathcal{P}$ be the plane that contains the points A, B, and D, and let $\overrightarrow{N}$ be a vector that is orthogonal to $\mathcal{P}$. Our goal is to show that the point C also lies on $\mathcal{P}$. One option is to actually find an equation for the plane $\mathcal{P}$, and then plug in C to see if it satisfies the equation. However, we don't really need to go all the way and find the equation of the plane.

The second (and quicker) option, is to notice that C lies on the plane if and only if the vector $\overrightarrow{AC}$ lies in the plane $\mathcal{P}$. On the other hand, $\overrightarrow{AC}$ lies in the plane $\mathcal{P}$ if and only if $\overrightarrow{AC}$ is orthogonal to $\overrightarrow{N}$. Since $\overrightarrow{N} = \overrightarrow{AB} \times \overrightarrow{AD}$, we just need to check if $(\overrightarrow{AB} \times \overrightarrow{AD}) \cdot \overrightarrow{AC} = 0$. Since,

$$(\overrightarrow{AB} \times \overrightarrow{AD}) \cdot \overrightarrow{AC} = \begin{vmatrix} 5 & 7 & 0 \\ 3 & 6 & -2 \\ 2 & 1 & 2 \end{vmatrix}$$

is just the triple product of the three vectors, we just have to check that that determinant is indeed 0. I'll leave that to you.