

Solutions to the second sample midterm 1

January 24, 2017

1. (a)

$$\overrightarrow{AB} = (-3, 2, 1) - (1, 0, 3) = (-4, 2, -2).$$

$$\|\overrightarrow{AB}\| = \sqrt{(-4)^2 + 2^2 + (-2)^2} = \sqrt{16 + 4 + 4} = \sqrt{24}.$$

Therefore, after normalizing \overrightarrow{AB} , we get

$$\frac{1}{\sqrt{24}}(-4, 2, -2).$$

(b) We did not explicitly cover midpoints in our class. Nevertheless, this problem should be fairly intuitive. To get the midpoint of A and B , you can just take the *average* of each of the components (try to convince yourself that this is true). With this in mind,

$$M = \left(\frac{-3+1}{2}, \frac{2+0}{2}, \frac{1+3}{2} \right) = (-1, 1, 1).$$

(c) In order to find the equation of any plane, we need a point, $P_0 = (x_0, y_0, z_0)$, on the plane, and a vector $\overrightarrow{N} = (a, b, c)$, that is *normal* to the plane. If we have these two pieces of information, then the equation of the plane is:

$$ax + by + cz = d$$

where $d = \overrightarrow{N} \cdot \overrightarrow{OP_0}$.

In this problem, we are given that we can use the point $M = (-1, 1, 1)$ for the point on the plane, and we can use $\overrightarrow{AB} =$

$(-4, 2, -2)$ for the normal vector. Note that $\overrightarrow{OM} \cdot \overrightarrow{AB} = 4 + 2 - 2 = 4$. Therefore, an equation of the plane is:

$$-4x + 2y - 2z = 4.$$

2. Let's compute the limit along two different paths. If (x, y) approaches $(0, 0)$ along the line where $x = 0$, we have

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ x=0}} \frac{xy}{x^2 + y^2} = \lim_{y \rightarrow 0} \frac{0 \cdot y}{0^2 + y^2} = \lim_{y \rightarrow 0} \frac{0}{y^2} = 0.$$

If (x, y) approaches $(0, 0)$ along the line where $y = 0$, we again find that the limit is 0 (I leave that to you). However, if (x, y) approaches $(0, 0)$ along the line $x = y$, we get:

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ x=y}} \frac{xy}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{x \cdot x}{x^2 + x^2} = \lim_{x \rightarrow 0} \frac{x^2}{2x^2} = \lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2}.$$

Since we get different values as (x, y) approaches $(0, 0)$ along two different paths, the limit **does not exist**.

3. (a) If \vec{u} is **any vector**, the magnitude of $\frac{\vec{u}}{\|\vec{u}\|} = 1$. Therefore,

$$\left\| \frac{-3\vec{v}}{\|\vec{v}\|} \right\| = |-3| \left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = |-3| \cdot 1 = 3.$$

- (b) Recall, the volume of the parallelepiped that is spanned by \vec{u} , \vec{v} , and \vec{w} is the absolute value of the **triple product** $\vec{u} \cdot (\vec{v} \times \vec{w})$. Therefore, the volume of the parallelepiped is

$$|(1, -2, 1) \cdot (1, 1, -1)| = |1 - 2 - 1| = |-2| = 2.$$

- (c) The area of the parallelogram spanned by $2\vec{v} + 3\vec{w}$ and $\vec{v} + \vec{w}$ is given by the magnitude of the cross product of the two vectors. Note that

$$(2\vec{v} + 3\vec{w}) \times (\vec{v} + \vec{w}) = 2(\vec{v} \times \vec{v}) + 2(\vec{v} \times \vec{w}) + 3(\vec{w} \times \vec{v}) + 3(\vec{w} \times \vec{w}).$$

Now, notice that if you take the cross product of any two *parallel* vectors, you get $\vec{0}$ ¹ Therefore,

$$\begin{aligned}(2\vec{v} + 3\vec{w}) \times (\vec{v} + \vec{w}) &= 2(\vec{v} \times \vec{w}) + 3(\vec{w} \times \vec{v}) \\ &= 2(\vec{v} \times \vec{w}) - 3(\vec{v} \times \vec{w}) . \\ &= -(\vec{v} \times \vec{w}).\end{aligned}$$

Finally,

$$\|(2\vec{v} + 3\vec{w}) \times (\vec{v} + \vec{w})\| = \|\vec{v} \times \vec{w}\| = 3.$$

- (d) First of all, since the angle between \vec{v} and \vec{w} is obtuse, $\vec{v} \cdot \vec{w} < 0$. Next, recall that

$$\|\text{proj}_{\vec{v}} \vec{w}\| = \frac{|\vec{w} \cdot \vec{v}|}{\|\vec{v}\|}.$$

Therefore, we have

$$5 = \frac{|\vec{v} \cdot \vec{w}|}{2},$$

and hence, $\vec{v} \cdot \vec{w} = -10$.

- (e) The strategy will be to find two vectors that are parallel to our plane, and cross them. The resulting vector will be orthogonal to our plane. Since the line is parallel to our plane, the direction vector $\vec{d} = (1, 2, 3)$ is parallel to our plane. Since our plane is perpendicular to the plane $x - y + z = 1$, it is *parallel* to the normal vector $(1, -1, 1)$ (draw a picture!). So the vector we are looking for is

$$(1, 2, 3) \times (1, -1, 1) = (5, 2, -3).$$

- (f) Let \vec{d} be the vector we are trying to find, i.e., \vec{d} is a vector that is parallel to the line of intersection of the two planes. Let

¹To see this, let \vec{u} and \vec{v} be parallel. Then $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$, where θ is the angle between \vec{u} and \vec{v} . Since the vectors are parallel, $\theta = 0$ or $\theta = \pi$. In either case, $\sin \theta = 0$, and therefore $\|\vec{u} \times \vec{v}\| = 0$. Since the zero vector is the only vector of length 0, we get $\vec{u} \times \vec{v} = \vec{0}$.

$\vec{N}_1 = (1, 1, 1)$ be a normal vector to the first plane, and let $\vec{N}_2 = (-1, 1, -1)$ be a normal vector to the second plane (Recall, we can get these vectors just by looking at the coefficients in front of x , y , and z in the equations of the planes). Since \vec{d} is parallel to the intersection of the two planes, \vec{d} must be parallel to both the plane $x + y + z = 1$ and the plane $-x + y - z = 0$. In order for \vec{d} to be parallel to $x + y + z = 1$, it must be orthogonal to the normal vector \vec{N}_1 . Similarly, since \vec{d} is parallel to the second plane, it must be orthogonal to the normal vector \vec{N}_2 . Since we need \vec{d} to be a vector orthogonal to \vec{N}_1 **and** \vec{N}_2 , we can take \vec{d} to be $\vec{N}_1 \times \vec{N}_2 = (-2, 0, 2)$.

4. Let's compute the level curves for these functions. For each function, we fix $f(x, y) = c$.

$$f_1(x, y) = c \implies x^3 - y = c \implies y = x^3 - c.$$

The level curves are therefore shifts of the graph $y = x^3$. So f_1 matches with Figure (b).

$$f_2(x, y) = c \implies xy = c \implies y = \frac{c}{x}.$$

The level curves are therefore generally shaped like the curve $y = \frac{1}{x}$, but stretched out by whatever c happens to be (except if $c = 0$, the level curve is the set of points $xy = 0 \implies x = 0$ or $y = 0$). So f_2 matches with Figure (d).

$$f_3(x, y) = c \implies x^2 - y^2 = c.$$

The level curves are therefore hyperbolas (see lecture notes from January 23). So f_3 matches with Figure (a).

$$f_4(x, y) = c \implies y - \ln(x) = c \implies y = \ln(x) + c.$$

The level curves are therefore shifts of the graph $y = \ln(x)$. So f_4 matches with Figure (c).