Solutions to the first sample midterm 2

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1. (a) We use the equation for the tangent plane:

$$z = f(3,4) + \frac{\partial f}{\partial x}(3,4)(x-3) + \frac{\partial f}{\partial y}(3,4)(y-4).$$

With the information we are given, the equation of the plane is:

$$z = 7 - 3(x-3) + 2(y-4).$$

(b) Let

$$L(x,y) = 7 - 3(x-3) + 2(y-4)$$

be the *linear approximation* of f near $(3,4)$. Then since $(3.1, 3.8)$ is near $(3,4)$, we have

$$f(3.1, 3.8) \approx L(3.1, 3.8) = 7 - 3(0.1) + 2(-0.2) = 6.3.$$

2. (a) We have

- $\dfrac{\partial f}{\partial x} = -2e^{-2x}\cos(Ay),$
- $\dfrac{\partial f}{\partial y} = -Ae^{-2x}\sin(Ay),$
- $\dfrac{\partial^2 f}{\partial x^2} = 4e^{-2x}\cos(Ay),$
- $\dfrac{\partial^2 f}{\partial y \partial x} = 2Ae^{-2x}\sin(Ay),$
- $\dfrac{\partial^2 f}{\partial y^2} = -A^2e^{-2x}\cos(Ay),$

- $\dfrac{\partial^2 f}{\partial x \partial y} = 2Ae^{-2x}\sin(Ay)$.

  Remark: It is not necessary to do both *mixed partials* if you correctly cite Clairaut's Theorem. Since $f(x, y)$ is the product of an exponential and a trigonometric function, it has continuous second partials and therefore the mixed partials must be equal.

(b) We are looking for all real numbers A so that

$$4e^{-2x}\cos(Ay) - A^2e^{-2x}\cos(Ay) = 0.$$

Since the equation is supposed to hold for all x, and y values, we can cancel out the $e^{-2x}\cos(Ay)$ terms to get

$$A^2 = 4 \implies A = \pm 2.$$

3. Let $\overrightarrow{c}(t) = (\sqrt{3+t}, 3 + \frac{1}{2}t)$ be the path function that describes the position of the beetle. Then the composition function $(T \circ \overrightarrow{c})(t)$ describes the temperature of the beetle at t seconds. Therefore, we are being asked to calculate $(T \circ \overrightarrow{c})'(t)$. By the chain rule, we have

$$D(T \circ \overrightarrow{c})(t) = [DT](\overrightarrow{c}(t))[D\overrightarrow{c}](t).$$

In this special case, this simplifies to

$$(T \circ \overrightarrow{c})'(t) = \nabla T(\overrightarrow{c}(t)) \cdot \overrightarrow{c}'(t).$$

After 6 seconds, the position of the beetle is

$$\overrightarrow{c}(6) = (3, 6).$$

Therefore,

$$(T \circ \overrightarrow{c})'(6) = \nabla T(3, 6) \cdot \overrightarrow{c}'(6).$$

We are given that $\nabla T(3, 6) = (12, 10)$, and a quick calculation shows $\overrightarrow{c}'(6) = (\frac{1}{6}, \frac{1}{2})$.

So we have

$$(T \circ \overrightarrow{c})'(6) = (12, 10) \cdot \left(\frac{1}{6}, \frac{1}{2}\right) = 7.$$

4. First observe that the domain D is both *closed and bounded*, and that h is a continuous function (it is just a polynomial). Therefore, the Extreme Value Theorem guarantees the existence of global extrema. To find the global extrema, we follow a three step process:
Step 1: Locate critical points on the interior of the domain.
Step 2: Locate any critical points on the boundary of the domain. Since the domain is a circle, it will be convenient to use the method of Lagrange multipliers for step 2, though for this particular function, it is better to use ad hoc methods. For the sake of practice, we will do both.

Step 3: Plug all of the critical points into the function and see which gives us the largest number and which gives us the smallest.

Step 1: (Find critical points on the interior) To find critical points on the interior, we need to solve for when $\nabla h = \overrightarrow{0}$. *Remark:* We should also take note of any points where h is not differentiable, but as we noted earlier, h is a polynomial and is therefore differentiable everywhere. Since

$$\nabla h = (4x + 8, 4y),$$

it is trivial to check that $\nabla f = \overrightarrow{0}$ only at the point $(-2, 0)$. Since this point is in the interior of our domain, it gives us a critical point.

Step 2: (Optimize h on the boundary using Lagrange multipliers) Now we must find critical points on the boundary, $x^2+y^2 = 25$, by optimizing the function $h(x, y) = 2x^2 + 2y^2 + 8x + 16$ *subject to the constraint* $g(x, y) = x^2 + y^2 = 25$. Note that $\nabla g = (2x, 2y) = (0, 0)$ only at the point $(x, y) = (0, 0)$. Since $(0, 0)$ is *not* on the constraint, we can disregard this point and move on to solving the Lagrange condition

$$\nabla h = \lambda \nabla g \implies (4x + 8, 4y) = \lambda(2x, 2y), \text{ and}$$
$$x^2 + y^2 = 25.$$

The first equation gives us

$$4x + 8 = \lambda 2x, \text{ and } 4y = \lambda 2y.$$

Using the equation $4y = \lambda 2y$, we see that either $y = 0$ (in which case it doesn't make sense to divide by y), or $\lambda = 2$.

Case 1: If $y = 0$, then $x^2 + y^2 = 25$ implies $x^2 = 25 \implies x = \pm 5$. Therefore, we pick up two points where h could potentially have an extreme: $(5, 0)$, $(-5, 0)$.

Case 2: If $\lambda = 2$, then $4x + 8 = \lambda 2x$ implies $4x + 8 = 4x \implies 8 = 0$. Since this makes no sense, λ cannot possibly be 2. Therefore, we get no critical points in this case.

Step 3: (Plug them in) Now we must plug in all of the points we found: $(5, 0)$, $(-5, 0)$, and $(-2, 0)$. If you plug all of these points into h, you will see that h has a maximum of 106 occurring at $(5, 0)$, and a minimum of 8 occurring at $(-2, 0)$.

Step 2: (Finding critical points on the boundary using ad hoc methods) On the boundary, $x^2 + y^2 = 25$, $h(x, y)$ becomes,

$$h(x, y) = 2(x^2 + y^2) + 8x + 16 = 50 + 8x + 16 = 8x + 66$$

where $-5 \leq x \leq 5$. This is a linear function with positive slope, so h has a maximum of 106 and a minimum of 26 when restricted to the circle $x^2 + y^2 = 25$. If I remember correctly, the author of this exam did not intend for the problem to be solved this way, and only a couple students saw this method as a solution.