# Solutions to the first sample midterm 2

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1. (a) We use the equation for the tangent plane:

$$z = f(1,3) + \frac{\partial f}{\partial x}(1,3)(x-1) + \frac{\partial f}{\partial y}(1,3)(y-3).$$

With the information we are given, the equation of the plane is:

$$z = 1 + (x-1) + 2(y-3).$$

(b) Let

$$L(x,y) = 1 + (x-1) + 2(y-3)$$

be the *linear approximation* of $f$ near $(1,3)$. Then since $(1.1, 2.9)$ is near $(1,3)$, we have

$$f(1.1, 2.9) \approx L(1.1, 2.9) = 1 + (0.1) + 2(-0.1) = 0.9.$$

2. In order to understand this question, be sure to review the geometric properties of the gradient.

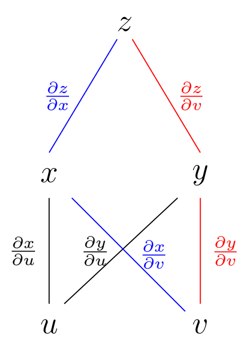
(a) To find the equation of any plane, we need a vector, $\overrightarrow{N}$ that is normal to the plane, and we need a point $P_0$ that lies on the plane. Since **gradients are orthogonal to level sets**, and since $g(1,2,3) = 1$, we know that $\nabla g(1,2,3)$ is **orthogonal to level surface,** $g(x,y,z) = 1$ **of** $g$. Therefore, we can use $\overrightarrow{N} = \nabla g(1,2,3) = (2,1,1)$. For the point on the plane, we can use $P_0 = (1,2,3)$. Finally, the plane is given by the equation:

$$2x + y + z = (2,1,1) \cdot (1,2,3) \implies 2x + y + z = 7.$$

(b) The maximum rate of change of $g$ at $(1,2,3)$ is $\|\nabla g(1,2,3)\| = \|(2,1,1)\| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$.

(c) Let $\overrightarrow{u} = \frac{1}{\sqrt{6}}(-1,2,-1)$ be the unit vector in the direction of $(-1,2,-1)$. Note that the directional derivative $[D_{\overrightarrow{u}}g](1,2,3) = \nabla g(1,2,3) \cdot \overrightarrow{u} = \frac{1}{\sqrt{6}}((2,1,1)\cdot(-1,2,-1)) < 0$ is negative. This means, by definition of the directional derivative, that $g$ is *decreasing* as you travel from $(1,2,3)$ in the direction $(-1,2,-1)$.

3. Since we only want a single partial derivative (and not the full matrix of partial derivatives), we can use the diagram shortcut. Note that you could also get the answer by using the matrix chain rule.
Refer to the following *dependency tree diagram:*




Since we want $\frac{\partial z}{\partial v}$, we need to locate $v$ in the diagram, and find all paths that go from $z$ to $v$. There are two paths that go from $z$ to $v$, namely the red path and the blue path. Now to get $\frac{partial z}{\partial v}$, we multiply the legs of each path together, and then add up the product that we get from each path. So

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

Now we just have to calculate:

$$\frac{\partial z}{\partial x} = \cos(y),$$

$$\frac{\partial x}{\partial v} = -4v,$$

$$\frac{\partial z}{\partial y} = -x\sin(y)$$

$$\frac{\partial y}{\partial v} = 1.$$

If we plug into the above formula, we get

$$\frac{\partial z}{\partial v} = -4v\cos(y) - x\sin(y).$$

Since the instructions explicitly say you can leave it in terms of $x, y, u$ and $v$. This answer is fine. If you would like to put it in terms of just $u$, and $v$, we get

$$\frac{\partial z}{\partial v} = -4v\cos(3u+v) - (u^2 - 2v^2)\sin((3u+v)).$$

4. **Step 1: (Find the critical points)** Recall that $f$ has a critical point at $(a, b)$ if either $\nabla f(a, b) = \overrightarrow{0}$ or $f$ is not differentiable at $(a, b)$. Since $f$ is a polynomial, it is differentiable everywhere and therefore we only need to find points where $\nabla = \overrightarrow{0}$. Since $\nabla f = (3x^2 - 3, 3y^2 - 6y)$, we must solve the system of equations

$$3x^2 - 3 = 0,$$

$$3y^2 - 6y = 0.$$

It is easy to see that any point $(x, y)$ where $x = 1$ or $x = -1$ and $y = 0$ or $y = 2$ is a critical point. So we have four critical points: $(1, 0)$, $(-1, 0)$, $(1, 2)$, $(-1, 2)$.

**Step 2: (Use the second derivative test to classify each critical point:)** Let's first calculate the *discriminant* of $f$.

$$\frac{\partial^2 f}{\partial x^2} = 6x$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = 6(y-1)$$

Therefore,

$$D(x,y) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial x^2}\right) - \left(\frac{\partial^2 f}{\partial y \partial x}\right) = 36x(y-1).$$

$(1,0)$**:** We have $D(1,0) = 36(1)(-1) < 0$. So $(1,0)$ is a Saddle point.

$(-1,0)$**:** We have $D(-1,0) = 36(-1)(-1) > 0$, and $\frac{\partial^2 f}{\partial x^2}(-1,0) = 6(-1) < 0$. So $f$ has a local maximum at $(-1,0)$.

$(1,2))$**:** We have $D(1,2) = 35(1)(1) > 0$, and $\frac{\partial^2 f}{\partial x^2}(1,2) = 6(1) > 0$. Therefore, fas has a local minimum at $(1,2)$.

$(-1,2)$**:** Finally, $D(-1,2) = 36(-1)(1) < 0$. Therefore $(-1,2)$ is a saddle point for $f$.