§1.3: 2, 3, 6, 7, 9, 13, 15, 16(b), 18, 20, 26, 27, 31*

$\boxed{2}$ (a) $\begin{vmatrix} 2 & -1 & 0 \\ 4 & 3 & 2 \\ 3 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 4 & 3 \\ 3 & 0 \end{vmatrix}$

$= 2(3-0) + (4-6) + 0$

$= 6 - 2$

$= \boxed{4}$

(b) $\begin{vmatrix} 36 & 18 & 17 \\ 45 & 24 & 20 \\ 3 & 5 & -2 \end{vmatrix} = 36\begin{vmatrix} 24 & 20 \\ 5 & -2 \end{vmatrix} - 18\begin{vmatrix} 45 & 20 \\ 3 & -2 \end{vmatrix} + 17\begin{vmatrix} 45 & 24 \\ 3 & 5 \end{vmatrix}$

$= 36\left(24(-2) - (20)(5)\right) - 18\left(45(-2) - (20)(3)\right) + 17\left(45(5) - 24(3)\right)$

$= \boxed{-27}$

(c) $\begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix} = 1\begin{vmatrix} 9 & 16 \\ 16 & 25 \end{vmatrix} - 18\begin{vmatrix} 4 & 16 \\ 9 & 25 \end{vmatrix} + 9\begin{vmatrix} 4 & 9 \\ 9 & 16 \end{vmatrix}$

$= (9(25) - 16^2) - 18((4)(25) - (16)(9)) + 9((4)(16) - 9^2)$

$= \boxed{-8}$

(d) $\begin{vmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{vmatrix} = 2\begin{vmatrix} 11 & 13 \\ 19 & 23 \end{vmatrix} - 3\begin{vmatrix} 7 & 13 \\ 17 & 23 \end{vmatrix} + 5\begin{vmatrix} 7 & 11 \\ 17 & 19 \end{vmatrix}$

$= 2((11)(23) - (13)(19)) - 3((7)(23) - (13)(17)) + 5((7)(19) - (11)(17))$

$= \boxed{-78}$

$\boxed{3}$ $\vec{a} = (1, -2, 1)$ $\vec{b} = (2, 1, 1)$

$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$

$= \vec{i}\left((-2)(1) - (1)(1)\right) - \vec{j}\left((1)(1) - (2)(1)\right) + \vec{k}\left((1)(1) - (2)(-2)\right)$

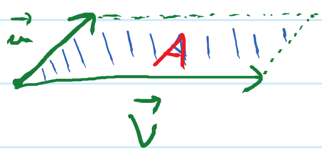
$= \vec{i}(-3) - \vec{j}(-1) + \vec{k}(5)$

$$= \boxed{(-3,1,5)}$$

$\boxed{\text{! Warning}}$ ① be careful with your arithmetic.
② Make sure you aren't calculating $\vec{b} \times \vec{a}$! Order matters!

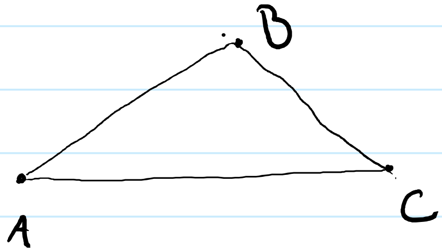
$\boxed{6}$ Use the cross product!

[Recall: $\|\vec{u} \times \vec{v}\|$ = The area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.

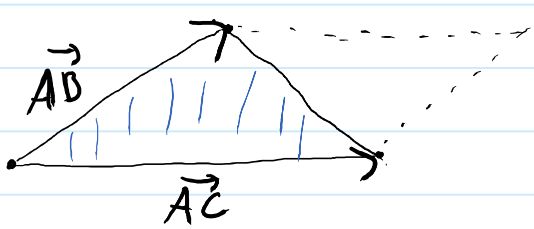


$A = \|\vec{u} \times \vec{v}\|$.

$\boxed{\text{Sol:}}$ Let $A=(0,0,0)$, $B=(1,1,1)$, $C=(0,-2,3)$
Draw the triangle.



Consider the vectors $\vec{AB}$, $\vec{AC}$.



The area of the Δ is ($\frac{1}{2}$ the area of the parallelogram spanned by $\vec{AB}$ and $\vec{AC}$) $= \frac{1}{2}\|\vec{AB} \times \vec{AC}\|$

Note: $\vec{AB} = \vec{OB} - \vec{OA} = (1,1,1) - (0,0,0) = \langle 1,1,1 \rangle$

$\vec{AC} = \vec{OC} - \vec{OA} = (0,-2,3) - (0,0,0) = (0,-2,3).$

Therefore, $\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 0 & -2 & 3 \end{vmatrix}$

$$= \vec{i} \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix}$$

$$= \vec{i}(5) - \vec{j}(3) + \vec{k}(-2)$$

$$= \boxed{(5, -3, -2)}$$

$$\|\vec{AB} \times \vec{AC}\| = \sqrt{5^2 + (-3)^2 + (-2)^2} = \sqrt{25+9+4}$$

$$= \boxed{\sqrt{38}}$$

So the area of the triangle is $\boxed{\frac{1}{2}\sqrt{38}}$

$\boxed{7}$ Let $\vec{u} = 2\vec{i} + \vec{j} - \vec{k} = (2, 1, -1)$

$\vec{v} = 5\vec{i} - 3\vec{k} = (5, 0, -3)$

$\vec{w} = \vec{i} - 2\vec{j} + \vec{k} = (1, -2, 1)$

Then the volume of the parallelopiped spanned by $\vec{u}$, $\vec{v}$, and $\vec{w}$ is

$$|(\vec{u} \times \vec{v}) \cdot \vec{w}| = \left| \begin{vmatrix} 2 & 1 & -1 \\ 5 & 0 & -3 \\ 1 & -2 & 1 \end{vmatrix} \right| = |-10| = \boxed{10}$$

$\boxed{9}$ Any vector that is orthogonal both $\vec{i}$ and $\vec{j}$ must be parallel to $\vec{i} \times \vec{j} = \vec{k}$.

So a vector $\vec{u}$ is orthogonal to both $\vec{i}$ and $\vec{j}$ iff $\vec{u} = c\vec{k}$ for some constant c.

13 $\vec{u} = (1,-2,1)$, $\vec{v} = (2,-1,2)$

$$\vec{u}+\vec{v} = (1+2,\ -2-1,\ 1+2) = \boxed{(3,-3,3)}$$

$$\vec{u}\cdot\vec{v} = (1)(2)+(-2)(-1)+(1)(2) = 2+2+2 = \boxed{6}$$

$$\|\vec{u}\| = \sqrt{1^2+(-2)^2+(1)^2} = \sqrt{1+4+1} = \boxed{\sqrt{6}}$$

$$\|\vec{v}\| = \sqrt{2^2+(-1)^2+2^2} = \sqrt{4+1+4} = \boxed{3}$$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & -1 & 2 \end{vmatrix} = \vec{i}\begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}$$

$$= \vec{i}\big((-2)(2)-(1)(-1)\big) - \vec{j}\big((1)(2)-(1)(2)\big) + \vec{k}\big((1)(-1)-(2)(-2)\big)$$

$$= \vec{i}(-3) - \vec{j}(0) + \vec{k}(3)$$

$$= \boxed{(-3,0,3)}$$

<u>Recall</u>: To find the eqn of a plane, you need:

① A point on the plane: $P_0 = (x_0, y_0, z_0)$, and

② A vector, $\vec{N} = (a,b,c)$ that is orthogonal to the plane. $\vec{N}$ is called the Normal vector.

If you have this information, the plane is given by

$$ax+by+cz = d \quad \text{where} \quad d = \vec{N}\cdot\vec{OP_0}$$

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15 (a) Here $\vec{N} = (1,1,1)$ and $P_0 = (1,0,0)$ so the eqn is:

$$1x+1y+1z = (1)(1)+(1)(0)+(1)(0)$$

or

$$\boxed{x+y+z=1}$$

(b) Here $\vec{N} = (1,2,3)$, $P_0 = (1,1,1)$ so an eqn for the plane is:

$$1x + 2y + 3z = (1)(1) + (2)(1) + 3(1) \quad \text{or}$$

$$\boxed{x + 2y + 3z = 6}$$

(c) Since the line is $\perp$ to the plane, the direction vector of $\vec{\ell}(t)$ is normal to the plane. So we can use $\vec{N} = (5,0,2)$

Since $\vec{\ell}(t) = (5,0,2)t + (3,-1,1)$, $(5,0,2)$ is a direction vector for $\vec{\ell}(t)$

For P_0, we can use $(5,-1,0)$ since the plane passes through this point. So an equation of the plane is:

$$5x + 0y + 2z = (5)(5) + (0)(-1) + (2)(0), \iff$$

$$\boxed{5x + 2z = 25}$$

(d) Same idea as part (c). Here, use $\vec{N} = (-1,-2,3)$, $P_0 = (2,4,-1)$.

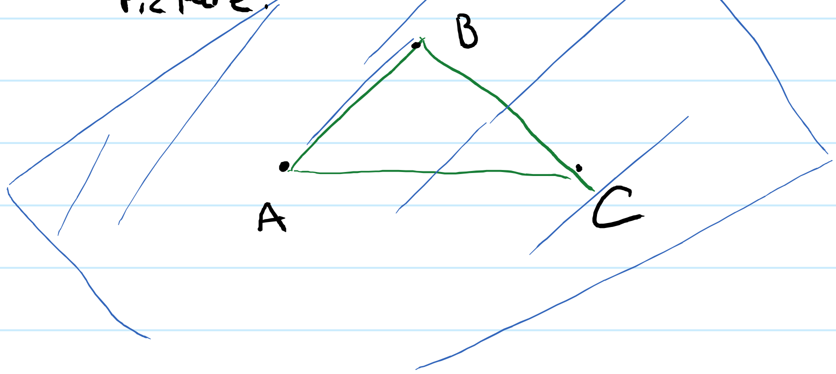
Then the equation is:

$$-1x - 2y + 3z = (-1)(2) + (-2)(4) + (3)(-1) \iff$$

$$\boxed{-x - 2y + 3z = -13}$$

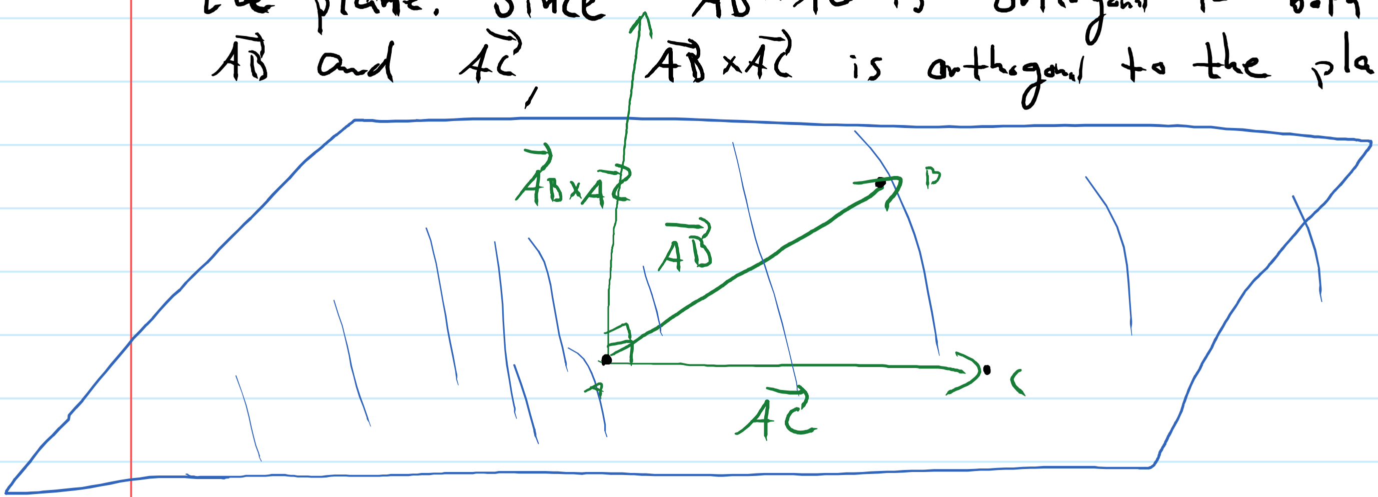
16(b): Let $A = (1,2,0)$, $B = (0,1,-2)$, $C = (4,0,1)$.

Picture:



We have 3 points on this plane, so for P_0, we can pick any one of these points. Let $\boxed{P_0 = (1,2,0)}$. The hard part is finding a vector normal to the plane.

Notice the vectors $\vec{AB}$ and $\vec{AC}$ are parallel to the plane. Since $\vec{AB} \times \vec{AC}$ is orthogonal to both $\vec{AB}$ and $\vec{AC}$, $\vec{AB} \times \vec{AC}$ is orthogonal to the plane.



So $\vec{N} = \vec{AB} \times \vec{AC}$. (Note: $\vec{AC} \times \vec{AB}$ also works).

$$\vec{AC} = \vec{OC} - \vec{OA} = (4,0,1) - (1,2,0) = (3,-2,1)$$
$$\vec{AB} = \vec{OB} - \vec{OA} = (0,1,-2) - (1,2,0) = (-1,-1,-2).$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ -1 & -1 & -2 \end{vmatrix} = (5,5,-5)$$

So eqn of the plane is:

$$5x + 5y - 5z = \underbrace{(5)(1) + (5)(2) + (-5)(0)}_{\vec{N} \cdot \vec{OP_0}}$$

$$\Leftrightarrow \quad 5x + 5y - 5z = 15$$
$$\Leftrightarrow \quad \boxed{x + y - z = 3}$$

18 Just plug in the points and see if the eqn holds:

$x+y+z=1$

(a) $0+0+0 \stackrel{?}{=} 1$ No! $(0,0,0)$ is not on the plane.

(b) $1+1-1 \stackrel{?}{=} 1$ Yes! $(1,1,-1)$ is on the plane.

(c) Yes $(-3,8,-4)$ is on the plane

(d) No $(1,2,-3)$ is not on the plane.

20 There are multiple ways to do this problem.

Method 1: Use "elimination": If we add the two equations, we get:

$$\left.\begin{aligned} x+2y+z&=0 \\ x-3y-z&=0 \end{aligned}\right\} \Rightarrow 2x-y=0 \Rightarrow \boxed{y=2x}$$

Substitute this back into one of the eqns. We get

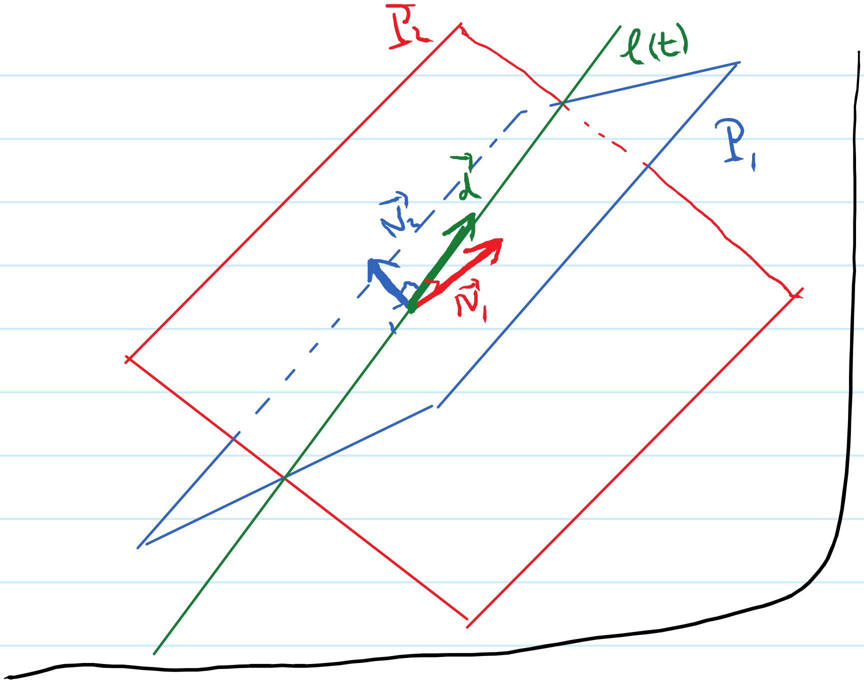
$$x+2(2x)+z=0 \Rightarrow 5x+z=0 \Rightarrow \boxed{z=-5x}.$$

If we let $x=t$ then, $y=2t$, $z=-5t$.

So we get the parametric equations

$$\boxed{\begin{aligned} x&=t \\ y&=2t \\ z&=-5t \end{aligned}}$$

Method 2 - Use geometry! Since the two planes are not parallel, they intersect in a line.



Let $\vec{N}_1 = (1,2,1)$ be a normal vector for the plane $x+2y+z=0$, and $\vec{N}_2 = (1,-3,-1)$ be a normal vector for $x-3y-z=1$.

Assume the line of intersection ℓ has the form $\vec{\ell}(t) = \vec{d}t + \vec{OP_0}$ where $\vec{d}$ is a vector parallel to ℓ and P_0 is a point on the line.

For P_0, we can use $(0,0,0)$ since this point is on both planes.

Notice that since ℓ is $\parallel$ to $x+2y+z=0$, $\vec{d} \perp \vec{N}_1$. Similarly $\vec{d} \perp \vec{N}_2$ since ℓ is $\parallel$ to $x-3y-z=0$.

$\therefore$ For $\vec{d}$, we can use $\vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 1 & -3 & -1 \end{vmatrix} = (1,2,-5)$.

So $\vec{\ell}(t) = (1,2,-5)t$

Note: This gives us the same answer!

26 Recall: $\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\| \sin(\theta)$ where θ is the smallest angle between $\vec{v}, \vec{w}$.

If $\|\vec{v} \times \vec{w}\| = \frac{1}{2}\|\vec{v}\|\,\|\vec{w}\|$, then

$$\sin(\theta) = \tfrac{1}{2} \Rightarrow \boxed{\theta = \tfrac{\pi}{6}}$$

So the smallest angle between $\vec{v}, \vec{w}$ is $\frac{\pi}{6}$

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27 Note: $\vec{v} = \vec{i} + \vec{j}$ $\quad \vec{w} = 2\vec{j} - \vec{k}$.

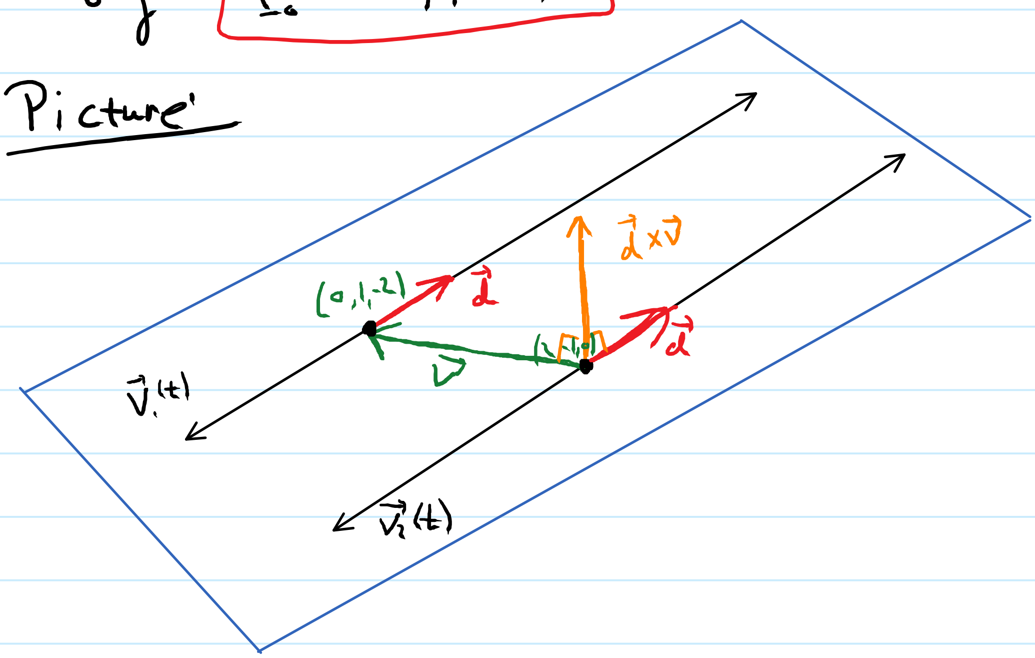
So
$$\begin{aligned}
\vec{v} \times \vec{w} &= (\vec{i} + \vec{j}) \times (2\vec{j} - \vec{k}) \\
&= 2(\vec{i} \times \vec{j}) - (\vec{i} \times \vec{k}) + 2(\vec{j} \times \vec{j}) - (\vec{j} \times \vec{k}) \\
&= 2\vec{k} - (-\vec{j}) + 2(\vec{0}) - (\vec{i}) \\
&= -\vec{i} + \vec{j} + 2\vec{k} \\
&= \boxed{(-1, 1, 2)}
\end{aligned}$$

31: We need: (1) A point on the plane, P_0
(2) a normal vector $\vec{N}$.

For P_0, use any point on either line since both lines are contained in the plane.

Say $\boxed{P_0 = (0, 1, -2)}$

Picture:



idea: we want to cross two vectors in the plane to get $\vec{N}$. The direction vector

$\vec{d} = (2,3,-1)$ is $\parallel$ to the plane,

and the points $(0,1,-2)$, $(2,-1,0)$ are both on the plane. So the difference vector

$$\vec{v} = (0,1,-2) - (2,-1,0)$$
$$= (-2,2,-2)$$

is $\parallel$ to the plane.

So let $\vec{N} = \vec{d} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ -2 & 2 & -2 \end{vmatrix} = (-4,6,10)$

Then an equation is:

$$-4x + 6y + 10z = (-4)(0) + (6)(1) + (10)(-2)$$
$$\Leftrightarrow \quad -4x + 6y + 10z = -14$$
$$\Leftrightarrow \quad \boxed{-2x + 3y + 5z = -7}$$