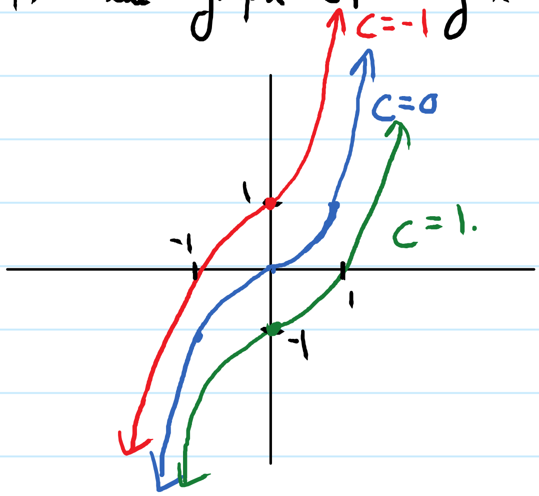
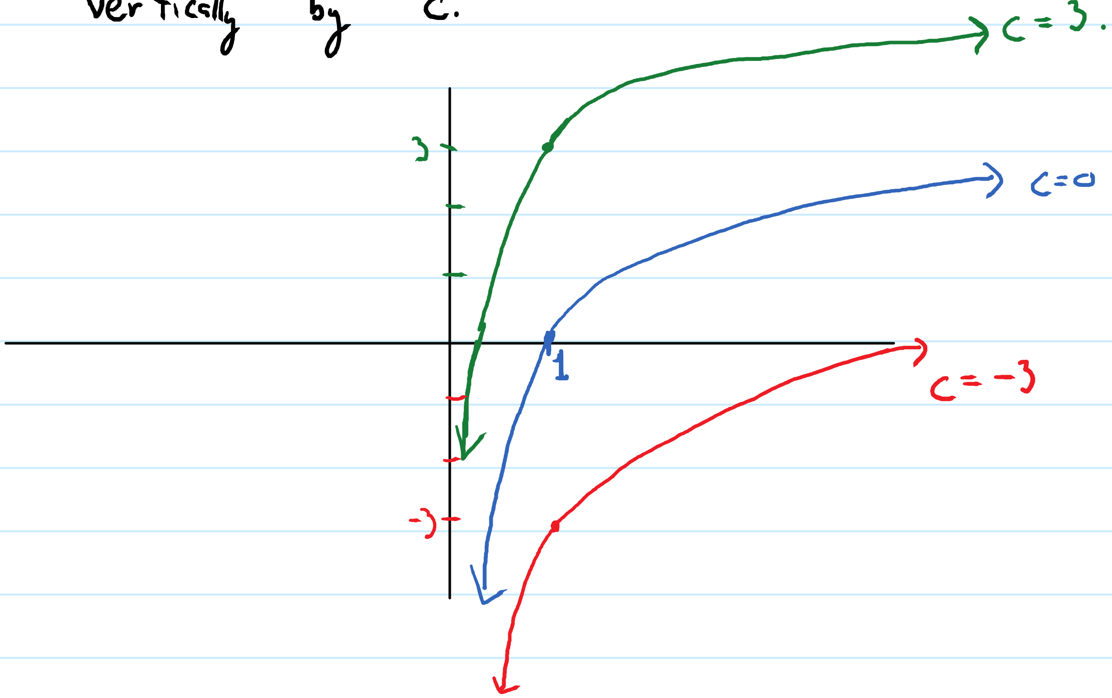
§ 2.1: 5(a,b), 6, 8, 13, 19

$\boxed{5a}$ $f(x,y) = x^3 - y$, $c = -1, 0, 1$

To draw level curves, set $f(x,y) = c$. We get $c = x^3 - y$, and so $y = x^3 - c$. The set of points (x,y) with $y = x^3 - c$ is the graph of $y = x^3$ shifted vertically by $-c$.



$\boxed{5b}$ $f(x,y) = y - 2\log(x)$, $c = -3, 0, 3$. If we set $z = f(x,y) = c$, we get $c = y - 2\log(x) \Rightarrow y = 2\log(x) + c$, which is just the graph of $y = 2\log(x)$, shifted vertically by c.



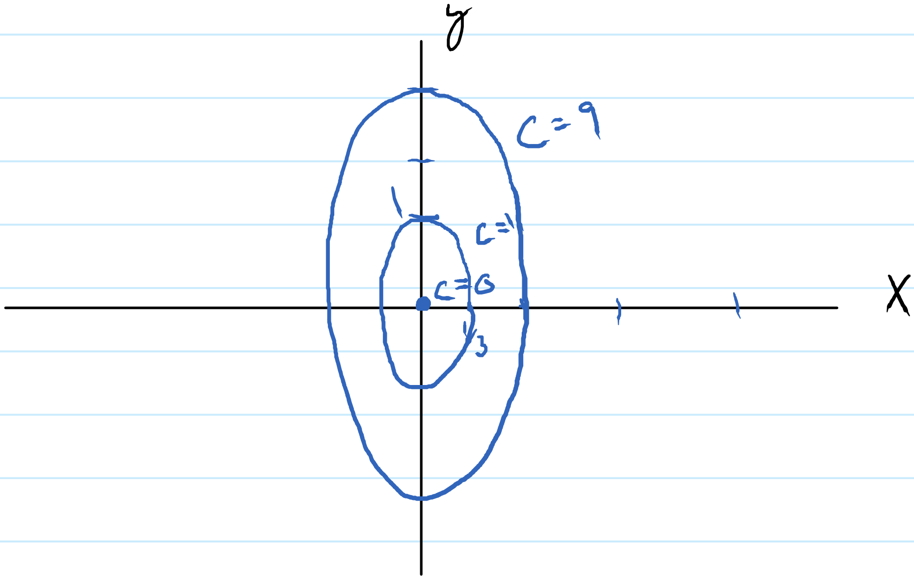
$\boxed{6a}$ Level curves of $f(x,y) = 9x^2 + y^2$.

$\underline{c=0:}$ $0 = 9x^2 + y^2 \Rightarrow (x,y) = (0,0)$

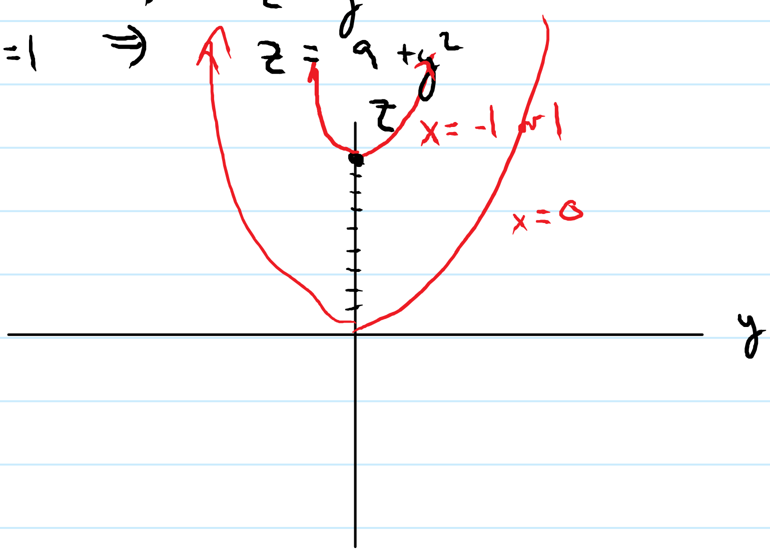
$\underline{c=1:}$ $1 = 9x^2 + y^2$

$\underline{c=9:}$ $9 = 9x^2 + y^2 \Rightarrow 1 = x^2 + \frac{y^2}{9}$

ellipses



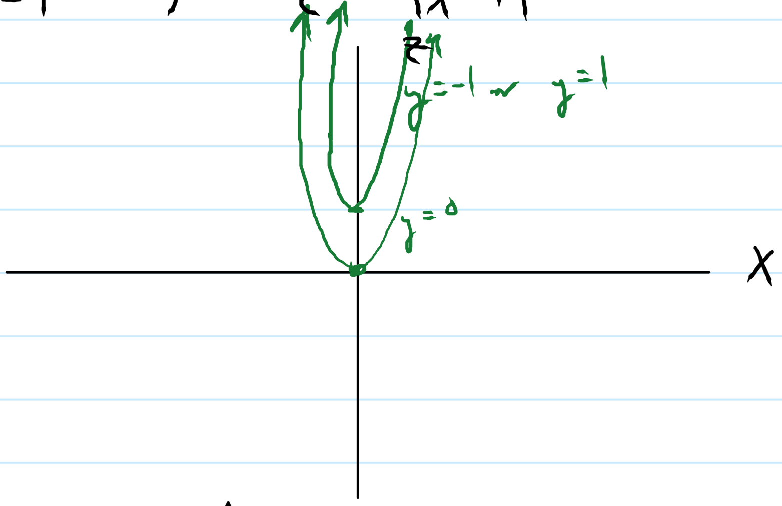
$\boxed{6b}$ $x=-1 \Rightarrow z = 9 + y^2$
$x=0 \Rightarrow z = y^2$
$x=1 \Rightarrow z = 9 + y^2$



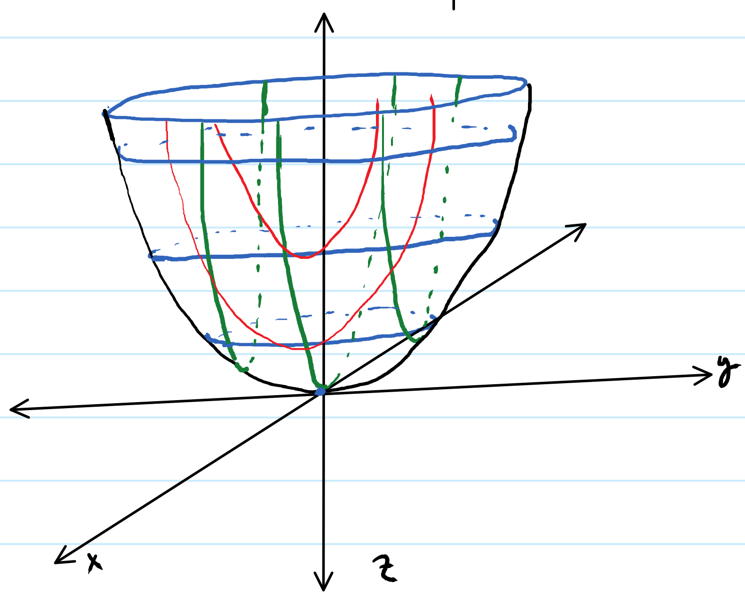
$\boxed{6c}$ $y=-1 \Rightarrow z = 9x^2+1$

$y=0 \Rightarrow z = 9x^2$

$y=1 \Rightarrow z = 9x^2+1$

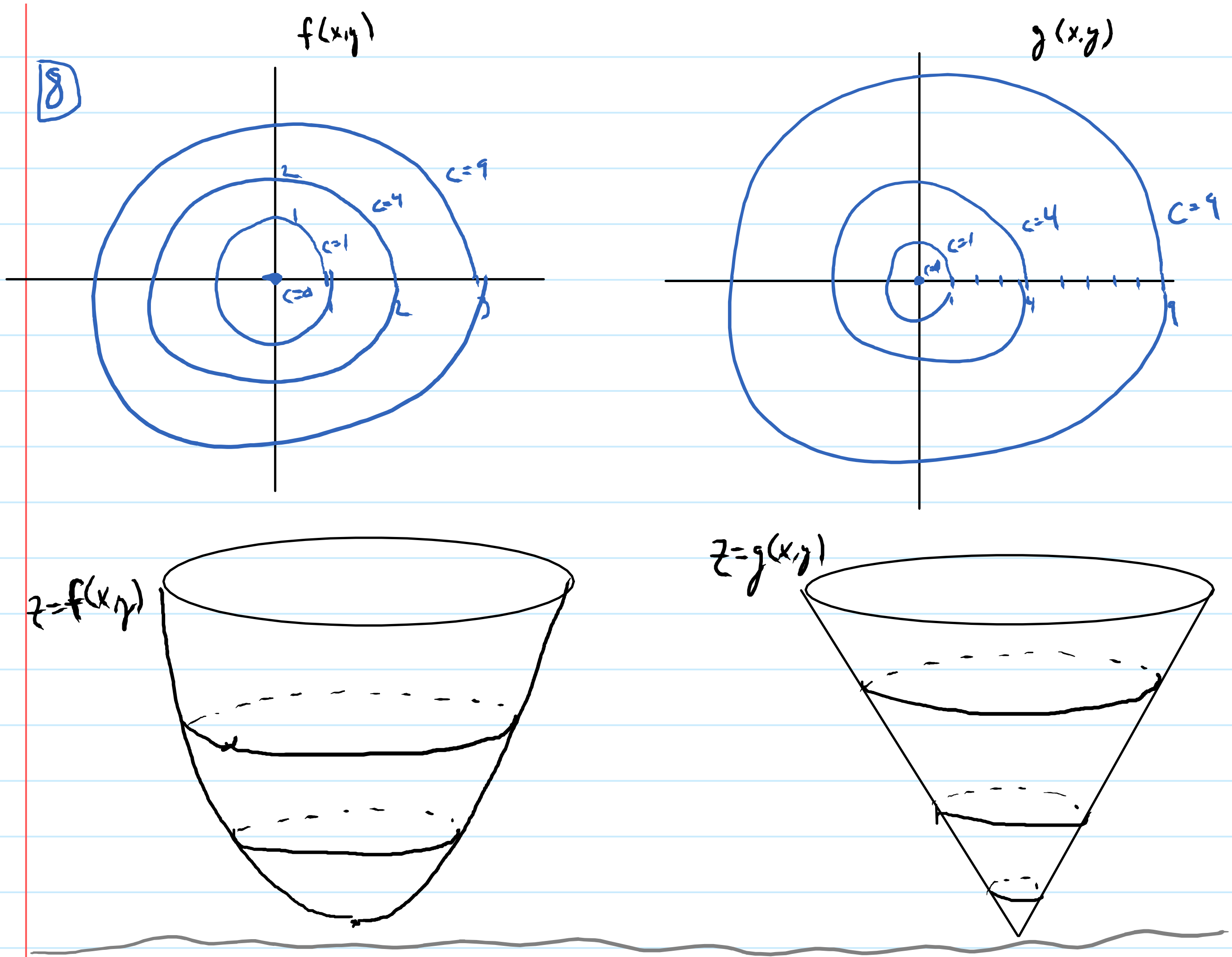


$\boxed{6d}$

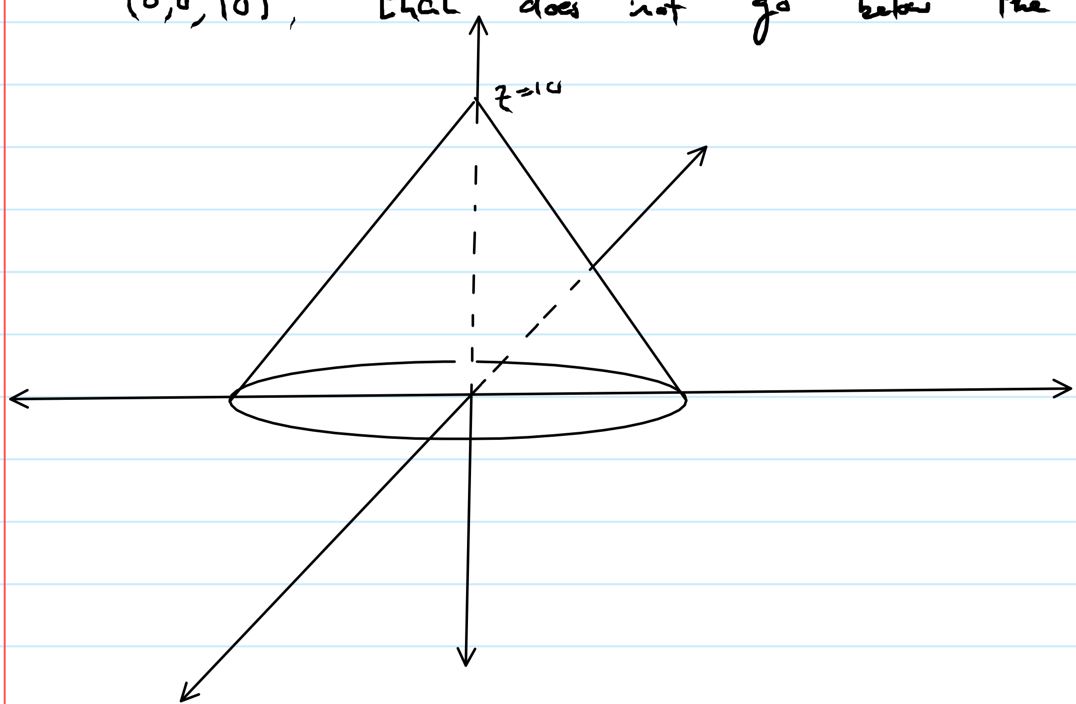


Visualize a "squished" bowl.

$\boxed{8}$



$\boxed{13}$ The level curves are circles centered at (0,0).
The graph is an upside down cone with vertex (0,0,10), that does not go below the xy-plane



$\boxed{19}$ $f(x,y,z) = -x^2 - y^2 - z^2.$

the level sets are

$$c = -x^2 - y^2 - z^2 \Rightarrow -c = x^2 + y^2 + z^2.$$

If $c < 0$, this is a sphere with radius $\sqrt{-c}$ centered at the origin.

§2.2: 2, 6, 7, 11 (Optional: 10, 12)

2 A fcn $f: \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous at (a,b) if and only if

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Since f is cont. and $\lim_{(x,y)\to(1,3)} f(x,y) = 5$, $f(1,3) = 5$.

6

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^6} & \text{if } (x,y) \neq \vec{0} \\ 0 & \text{if } (x,y) = \vec{0}. \end{cases}$$

a $$\lim_{\substack{(x,y)\to(0,0) \\ x=0}} f(x,y) = \lim_{y\to 0} \frac{0\cdot y^3}{0^2+y^6} = \lim_{y\to 0} \frac{0}{y^6} = \lim_{y\to 0} 0 = \boxed{0}$$

b $$\lim_{\substack{(x,y)\to(0,0) \\ x=y^3}} f(x,y) = \lim_{y\to 0} \frac{y^3\cdot y^3}{(y^3)^2+y^6} = \lim_{y\to 0} \frac{y^6}{2y^6} = \boxed{\tfrac{1}{2}}$$

c By parts a and b, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Therefore, f cannot be cont. at $(0,0)$.

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7 $$\lim_{h\to 0} \frac{1}{h}\left(f(1,2+h,3) - f(1,2,3)\right) = \lim_{h\to 0} \frac{1}{h}\left(\left(\frac{e^{1+(2+h)}}{1+3^2}\right) - \left(\frac{e^{1+2}}{1+3^2}\right)\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{e^{3+h}}{9} - \frac{e^3}{9}\right)$$

$$= \lim_{h\to 0} \frac{e^3}{9}\left(\frac{e^h - 1}{h}\right)$$

$$= \frac{e^3}{9} \lim_{h\to 0} \frac{e^h-1}{h}$$

$$= \boxed{\frac{e^3}{9}}$$

Rmk: ① The limit $\lim_{h\to 0}\left(\frac{e^h-1}{h}\right)=1$ is used to prove $\frac{d}{dx}(e^x)=e^x$. Try computing it without L'Hôpital's rule since that argument is circular.

② This limit is the definition $\frac{\partial f}{\partial y}(1,2,3)$.

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11 a Let $t=xy$. Then as $(x,y)\to(0,0)$, $t\to 0$.

So $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} = \lim_{t\to 0}\frac{\sin(t)}{t} \underset{L'H.}{=} \boxed{1}$

b Let $s=xyz$. Then as $(x,y,z)\to(0,0,0)$, $s\to 0$.

So $\lim_{(x,y,z)\to(0,0,0)} \frac{\sin(xyz)}{xyz} = \lim_{s\to 0}\frac{\sin(s)}{s} \underset{L'H}{=} \boxed{1}$

c $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+3y^2}{x+1} = \frac{0^2+3\cdot 0^2}{0+1} = \boxed{0}$

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10 a $\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x+1} = \frac{e^0}{0+1} = \boxed{1}$

b $\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{\cos x - 1 - \frac{x^2}{2}}{x^4+y^4} = \lim_{y\to 0}\frac{\cos(0)-1-\frac{0^2}{2}}{0^2+y^2} = \lim_{y\to 0}\frac{1-1}{y^2} = \lim_{y\to 0} 0$

$= \boxed{0}$

$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{\cos(x)-1-\frac{x^2}{2}}{x^4+y^4} = \lim_{x\to 0}\frac{\cos(x)-1-\frac{x^2}{2}}{x^4} = \lim_{x\to 0}\frac{-\sin(x)-x}{4x^3}$

$= \lim_{x\to 0}\frac{-\cos(x)-1}{12x^2}$

$= \lim_{x\to 0}\frac{\sin(x)}{24x}$

$= \lim_{x\to 0}\frac{\cos(x)}{24}$

$= \frac{\cos(0)}{24}$

$= \boxed{\frac{1}{24}}$

L'H four times

Since the limits along two different paths do not match, the limit DNE c $\lim\limits_{\substack{(x,y)\to(0,0)\\ x=0}} \dfrac{(x-y)^2}{x^2+y^2} = \lim\limits_{y\to 0} \dfrac{(0-y)^2}{0^2+y^2} = \lim\limits_{y\to 0} \dfrac{y^2}{y^2} = \boxed{1}$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ x=y}} \dfrac{(x-y)^2}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{(x-x)^2}{2x^2} = \lim\limits_{x\to 0} \dfrac{0}{2x^2} = \boxed{0}$

Since the limits don't match along the two different paths, the limit DNE.

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12 a $\lim\limits_{x\to 0} \dfrac{\sin 2x - 2x}{x^3} = \lim\limits_{x\to 0} \dfrac{2\cos 2x - 2}{3x^2}$

$= \lim\limits_{x\to 0} \dfrac{-4\sin 2x}{6x}$

$= \lim\limits_{x\to 0} \dfrac{-8\cos 2x}{6}$

$= \dfrac{-8\cos(0)}{6}$

$= \boxed{\dfrac{-4}{3}}$

b $\lim\limits_{\substack{(x,y)\to(0,0)\\ x=0}} \dfrac{\sin 2x - 2x + y}{x^3+y} = \lim\limits_{y\to 0} \dfrac{\sin(0) - 2(0) + y}{0+y} = \lim\limits_{y\to 0} \dfrac{y}{y} = \boxed{1}$

not equal

$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=0}} \dfrac{\sin 2x - 2x + y}{x^3+y} = \lim\limits_{x\to 0} \dfrac{\sin 2x - 2x}{x^3} = \boxed{\dfrac{-4}{3}}$

The limit DNE.

c We could use cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. and then use Squeeze Thm. Since we haven't talked about this yet, I'll avoid this.

Observation 1: For all z, $0 \le |\cos z| \le 1$.

Observation 2: Since $y^2 \ge 0$, $x^2+y^2 \ge x^2$, and so $\dfrac{1}{x^2+y^2} \le \dfrac{1}{x^2}$.

We therefore have the inequality

$$0 \leq \left| \frac{\cos(z)}{x^2+y^2} \right| \leq \left| \frac{1}{x^2} \right|$$

$$\Rightarrow \quad 0 \leq \left| \frac{2x^3y\cos(z)}{x^2+y^2} \right| \leq \left| \frac{2x^3y}{x^2} \right| = |2y|.$$

$$\Rightarrow 0 = \lim_{(x,y,z)\to(0,0,0)} 0 \leq \lim_{(x,y,z)\to(0,0,0)} \left| \frac{2x^3y\cos(z)}{x^2+y^2} \right| \leq \lim_{(x,y,z)\to(0,0,0)} |2y| = 0$$

$$\Rightarrow \lim_{(x,y,z)\to(0,0,0)} \frac{2x^3y\cos(z)}{x^2+y^2} = 0$$ by the Squeeze Thm.

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§2.3: 1, 3, 5, 6, 16, 22:

1a $f(x,y) = xy$

$$\boxed{\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x}$$

1b $f(x,y) = e^{xy}$:
$$\boxed{\frac{\partial f}{\partial x} = ye^{xy} \qquad \frac{\partial f}{\partial y} = xe^{xy}}$$

1c $f(x,y) = x\cos(x)\cos(y)$:
$$\boxed{\frac{\partial f}{\partial x} = \cos(x)\cos(y) - x\sin(x)\cos(y) \qquad \frac{\partial f}{\partial y} = -x\cos(x)\sin(y)}$$

1d $f(x,y) = (x^2+y^2)\log(x^2+y^2)$:
$$\frac{\partial f}{\partial x} = 2x\log(x^2+y^2) + (x^2+y^2)\left(\frac{1}{x^2+y^2}\right)\cdot(2x)$$
$$= 2x\log(x^2+y^2) + 2x$$
$$\frac{\partial f}{\partial y} = 2y\log(x^2+y^2) + 2y$$

3a $w = xe^{x^2+y^2}$:
$$\frac{\partial w}{\partial x} = e^{x^2+y^2} + 2x^2e^{x^2+y^2}$$
$$\frac{\partial w}{\partial x} = 2xy\,e^{x^2+y^2}$$

3b $w = \dfrac{x^2+y^2}{x^2-y^2}$:
$$\frac{\partial w}{\partial x} = \frac{2x(x^2-y^2) - 2x(x^2+y^2)}{(x^2-y^2)^2}$$

$$\boxed{\frac{\partial w}{\partial y} = \frac{2y(x^2-y^2) + 2y(x^2+y^2)}{(x^2-y^2)^2}}$$

3c $w = e^{xy}\log(x^2+y^2)$:

$$\boxed{\begin{aligned} \frac{\partial w}{\partial x} &= ye^{xy}\log(x^2+y^2) + \frac{2xe^{xy}}{x^2+y^2} \\ \frac{\partial w}{\partial y} &= xe^{xy}\log(x^2+y^2) + \frac{2ye^{xy}}{x^2+y^2} \end{aligned}}$$

3d $w = \frac{x}{y}$:

$$\boxed{\begin{aligned} \frac{\partial w}{\partial x} &= \frac{1}{y} \\ \frac{\partial w}{\partial y} &= \frac{-x}{y^2} \end{aligned}}$$

3e $w = \cos(ye^{xy})\sin(x)$:

$$\boxed{\begin{aligned} \frac{\partial w}{\partial x} &= -y^2e^{xy}\sin(ye^{xy})\sin(x) + \cos(ye^{xy})\cos(x) \\ \frac{\partial w}{\partial y} &= -\sin(x)\sin(ye^{xy})\cdot(e^{xy} + xye^{xy}) \end{aligned}}$$

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5 Recall: Eqn of tan. plane at $x=a$, $y=b$ is

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

Here, $a=3$, $b=1$, $z=f(x,y)=x^2+y^3$. So $f(a,b)=10$.

$$\frac{\partial f}{\partial x} = 2x \Rightarrow \frac{\partial f}{\partial x}(3,1) = 6$$

$$\frac{\partial f}{\partial y} = 3y^2 \Rightarrow \frac{\partial f}{\partial y}(3,1) = 3$$

Eqn of plane:

$$\boxed{z = 10 + 6(x-3) + 3(y-1)}$$

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6 $f(0,0) = e^0 = 1$

$$\frac{\partial f}{\partial x} = e^{x+y} \Rightarrow \frac{\partial f}{\partial x}(0,0) = e^0 = 1$$

$$\frac{\partial f}{\partial y} = e^{x+y} \Rightarrow \frac{\partial f}{\partial y}(0,0) = e^0 = 1$$

Eqn of plane:

$$\boxed{z = 1 + x + y}$$

$\boxed{16a}$ Let $f(x,y) = (xe^y)^8$ (which is diff'ble).
Use the tan plane at $(1,0)$
Then $(0.99e^{0.02})^8 \approx f(1,0) + \frac{\partial f}{\partial x}(1,0)(0.99-1) + \frac{\partial f}{\partial y}(1,0)(0.02-0)$

$f(1,0) = 1$, $\quad \frac{\partial f}{\partial x} = 8(xe^y)^7 \cdot e^y \Rightarrow \frac{\partial f}{\partial x}(1,0) = 8(1)^7 \cdot 1$
$= 8$

$\frac{\partial f}{\partial y} = 8(xe^y)^7 \cdot xe^y \Longrightarrow \frac{\partial f}{\partial y} = 8$

So $(0.99e^{0.02})^8 \approx 1 + 8(0.99-1) + 8(0.02)$
$= 1 - 0.08 + 0.16$
$= \boxed{0.92}$

$\boxed{16b}$ Let $f(x,y) = x^3 + y^3 - 6xy$ and use the tan plane at $(1,2)$.

$f(1,2) = 1 + 8 - 12 = -3$
$\frac{\partial f}{\partial x}(x,y) = 3x^2 - 6y \Rightarrow \frac{\partial f}{\partial x}(1,2) = 3 - 12 = -9$
$\frac{\partial f}{\partial y} = 3y^2 - 6x \Rightarrow \frac{\partial f}{\partial y}(1,2) = 12 - 6 = 6.$

$f(0.99, 2.01) \approx -3 - 9(0.99-1) + 6(2.01-2)$
$= -3 - 9(-0.01) + 6(0.01)$
$= -3 + 0.09 + 0.06$
$= \boxed{-2.85}$

$\boxed{16c}$ Let $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ and use linear approx. at $(4,4,2)$

22 $\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$

$= \lim_{h\to 0} \frac{1}{h}\left(\frac{h^2 \cdot 0^4}{h^4+0^8} - 0\right)$

$= \lim_{h\to 0} \frac{1}{h}(0-0)$

$= \lim_{h\to 0} 0$

$= \boxed{0}$

By a similar calculation, $\frac{\partial f}{\partial y}(0,0) = 0$.

22b f is not continuous at $(0,0)$ because

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist:

To see this, calculate the limit along the paths $x=0$ and $x=y^2$. (See §2.2, Problem 6 for help).