Homework 4

§2.3: 9, 10, 11, 14, 22, 24, 25:

$\boxed{9}\boxed{a}$ $f(x,y) = (x,y)$

$$[Df](x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\boxed{9b}$ $f(x,y) = (xe^y + \cos(y),\ x,\ x + e^y)$

$$[Df](x,y) = \begin{bmatrix} e^y & xe^y - \sin(y) \\ 1 & 0 \\ 1 & e^y \end{bmatrix}$$

$\boxed{9c}$ $f(x,y,z) = (x + e^z + y,\ yx^2)$

$$[Df](x,y,z) = \begin{bmatrix} 1 & 1 & e^z \\ 2xy & x^2 & 0 \end{bmatrix}$$

$\boxed{9d}$ $f(x,y) = (xye^{xy},\ x\sin(y),\ 5xy^2)$

$$[Df](x,y) \begin{bmatrix} ye^{xy} + xy^2e^{xy} & xe^{xy} + x^2ye^{xy} \\ \sin(y) & x\cos(y) \\ 5y^2 & 10xy \end{bmatrix}$$

$\boxed{10a}$ $f(x,y) = (e^x,\ \sin(xy))$

$$[Df](x,y) = \begin{bmatrix} e^x & 0 \\ y\cos(xy) & x\cos(xy) \end{bmatrix}$$

$f(x,y,z) = (x-y,\ y+z)$

$$[Df](x,y,z) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

10c $f(x,y) = (x+y,\ x-y,\ xy)$

$$[Df](x,y) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ y & x \end{bmatrix}$$

10d $f(x,y,z) = (x+z,\ y-5z,\ x-y)$

$$[Df](x,y,z) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & -1 & 0 \end{bmatrix}$$

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11 Recall: Eqn of tan. plane at (a,b) is

$$\boxed{z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).}$$

We are looking for a point (a,b) with $\frac{\partial f}{\partial x}(a,b) = 2$, $\frac{\partial f}{\partial y}(a,b) = 4$.

$$\frac{\partial f}{\partial x} = 2x - 2y$$
$$\frac{\partial f}{\partial y} = -2x + 4y$$

So we must solve the system of equations

$$\left.\begin{aligned} 2x - 2y &= 2 \\ -2x + 4y &= 4 \end{aligned}\right\} \xRightarrow{\text{add the eqns}} 2y = 6 \Rightarrow \boxed{y = 3}$$

Then $2x - 2y = 2 \Rightarrow 2x - (2)(3) = 2 \Rightarrow \boxed{x = 4}$

So $(a,b) = (3,4)$. Since $f(3,4) = 9 - 2(3)(4) + 2(4^2)$
$= 9 - 24 + 32$
$= 17$

So the eqn of the plane is

$$\boxed{z = 17 + 2(x-4) + 4(y-3)}$$

14 The tan. plane to f at (0,0) is

$$z = 0$$

(Try to see this geometrically!).

Since: $g(0,0) = 0$

$$\frac{\partial g}{\partial x}(0,0) = (-2x + y^3)\Big|_{\substack{x=0\\y=0}} = 0$$

$$\frac{\partial g}{\partial y}(0,0) = (-2y + 3xy^2)\Big|_{\substack{x=0\\y=0}} = 0,$$

the tan. plane to g at (0,0) is also

$$z = 0.$$

So it makes sense to say f, g are tangent at (0,0)

24 $h(x,y,z) = (x+z)e^{x-y}$

$$\nabla h = \left(e^{x-y} + (x+z)e^{x-y},\ -(x+z)e^{x-y},\ e^{x-y}\right)$$

$$\nabla h(1,1,1) = \left(e^{1-1} + (1+1)e^{1-1},\ -(1+1)e^{1-1},\ e^{1-1}\right)$$

$$= (1+2, -2, 1)$$

$$\boxed{= (3, -2, 1)}$$

25 $f(x,y,z) = x^2 + y^2 - z^2.$

$$\nabla f = (2x, 2y, -2z)$$

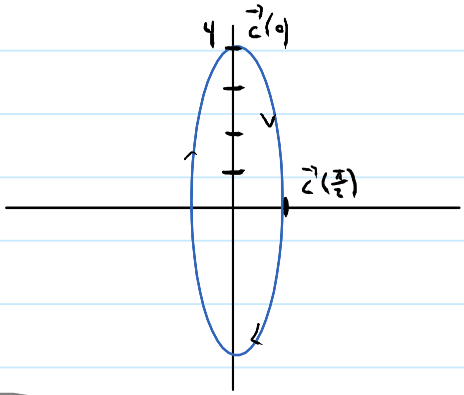
$$\boxed{\nabla f(0,0,1) = (0,0,-2)}$$

§2.4: 1, 5, 6, 7, 9, 17, 21, 25:

1 $\vec{c}(t) = (\sin(t), 4\cos(t)) \quad 0 < t \leq 2\pi$

$$\left.\begin{aligned} \vec{c}(0) &= (0,4) \\ \vec{c}(\tfrac{\pi}{2}) &= (1,0) \\ \vec{c}(\pi) &= (0,-4) \\ \vec{c}(\tfrac{3\pi}{2}) &= (-1,0) \\ \vec{c}(2\pi) &= (0,4). \end{aligned}\right\}$$

$\vec{c}$ parametrizes an ellipse in a clock-wise orientation



5 $\vec{c}(t) = (2\cos(t), 2\sin(t))$ (see class notes on Feb. 1st)

6a We already know how to do this:

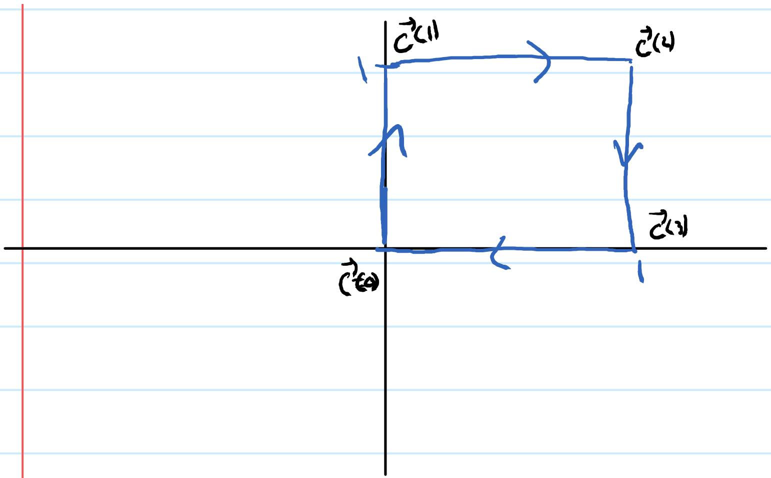
$$\boxed{\vec{\ell}(t) = (1,2,3) + t(-3,-2,4)}$$

6b If $x = t$, then $y = f(x) = t^2$, so we can use

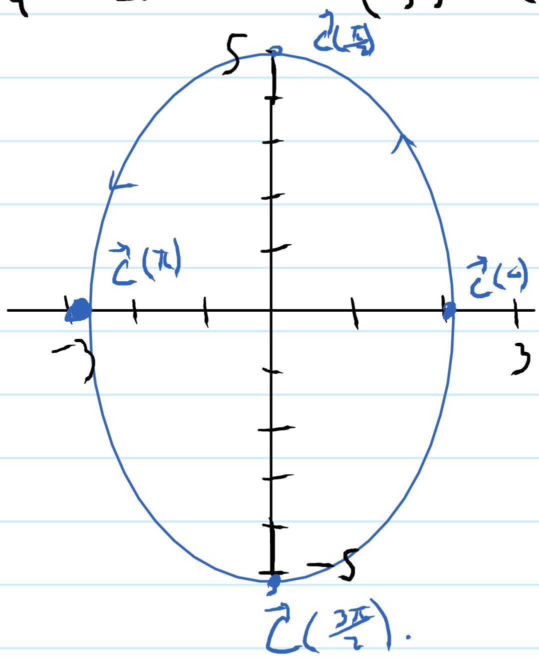
$$\boxed{\vec{c}(t) = (t, t^2)}$$

6c We use a piecewise fcn to parametrize each side:

$$\vec{c}(t) = \begin{cases} (0,t) & 0 \leq t \leq 1 \\ (t-1, 1) & 1 < t \leq 2 \\ (1, 1-(t-2)) & 2 < t \leq 3 \\ (1-(t-3), 0) & 3 < t \leq 4 \end{cases}$$



$\boxed{6d}$ $\frac{x^2}{9} + \frac{y^2}{25} = 1 \Rightarrow \left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$



$\vec{c}(t) = (3\cos(t), 5\sin(t))$

(Compare to $\vec{c}(t) = (\cos(t), \sin(t))$)

7 $\vec{c}(t) = (6t, 3t^2, t^3)$

$\boxed{\vec{c}'(t) = (6, 6t, 3t^2)}$

$$\boxed{\begin{aligned} &9\quad \vec{r}(t) = (\cos^2(t), 3t - t^3, t) \\ &\vec{r}'(t) = (-2\cos(t)\sin(t), 3 - 3t^2, 1) \end{aligned}}$$

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17 Recall: The eqn of the line tangent to $\vec{c}(t)$ at $t = t_0$ is

$$\boxed{\vec{\ell}(t) = \vec{c}(t_0) + (t - t_0)\vec{c}'(t_0).}$$

$\vec{c}(1) = (\sin(3), \cos(3), 2).$

$\vec{c}'(1) = (3\cos 3t, -3\sin 3t, 5t^{3/2})\Big|_{t=1} = (3\cos(3), -3\sin(3), 5).$

So $\boxed{\vec{\ell}(t) = (\sin(3), \cos(3), 2) + (t-1)(3\cos(3), -3\sin(3), 5).}$

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21 We want $\vec{\ell}(1)$ where $\vec{\ell}(t) = \vec{c}(0) + (t-0)\vec{c}'(0)$

$\vec{c}(0) = (4e^0, 6(0)^4, \cos(0)) = (4, 0, 1)$

$\vec{c}'(t) = (4e^t, 24t^3, -\sin(t))$

$\vec{c}'(0) = (4, 0, -\sin(0)) = (4, 0, 0).$

So $\vec{\ell}(t) = (4, 0, 1) + t(4, 0, 0)$

$\vec{\ell}(1) = \boxed{(8, 0, 1)}$

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25 9 $\vec{c}(t) = (t^3, t^2, 2t)$, $\quad f(x, y, z) = (x^2 - y^2, 2xy, z^2)$

$(f \circ c)(t) = ((t^3)^2 - (t^2)^2, 2t^3t^2, (2t)^2)$

$\boxed{= (t^6 - t^4, 2t^5, 4t^2)}$

$\boxed{25b}$ $(f\circ\vec{c})(1) = (0, 2, 4)$

$(f\circ\vec{c})'(t) = (6t^5 - 4t^3, 10t^4, 8t)$

$(f\circ\vec{c})'(1) = (2, 10, 8)$

So the line is given by

$$\boxed{\vec{\ell}(t) = (0,2,4) + (t-1)(2,10,8)}$$