§ 2.5: 3, 7, 8, 9, 10, 12, 13, 17, 18, 19, 34, 36:

$\boxed{3}$ ⓐ $f(x,y) = xy \qquad \vec{c}(t) = (e^t, \cos(t))$

- $(f\circ\vec{c})(t) = e^t\cos(t)$

  $(f\circ\vec{c})'(t) = e^t\cos(t) - e^t\sin(t)$
- $(f\circ\vec{c})'(t) = (\nabla f)(\vec{c}(t)) \cdot \vec{c}\,'(t)$.

  $\nabla f(\vec{c}(t)) = (y, x)\Big|_{\vec{c}(t)} = (\cos(t), e^t)$

  $\vec{c}\,'(t) = (e^t, -\sin(t))$.

chain rule: $(f\circ\vec{c})'(t) = (\nabla f)(\vec{c}(t)) \cdot \vec{c}\,'(t)$

$= (\cos(t), e^t) \cdot (e^t, -\sin(t))$

$= e^t\cos(t) - e^t\sin t$ ✓

$\boxed{3b-d:}$ Similar to 3a.

$\boxed{7}$ $f(u,v) = (\tan(u-1) - e^v,\ u^2 - v^2)$

$g(x,y) = (e^{x-y},\ x-y)$.

$$(f\circ g)(x,y) = f(g(x,y))$$
$$= f(e^{x-y}, x-y)$$
$$= (\tan(e^{x-y}-1) - e^{x-y},\ (e^{x-y})^2 - (x-y)^2)$$

$$[D(f\circ g)] = \begin{bmatrix} e^{x-y}\sec^2(e^{x-y}-1) - e^{x-y} & -e^{x-y}\sec^2(e^{x-y}-1) + e^{x-y} \\ 2e^{2x-2y} - 2(x-y) & -2e^{2x-2y} + 2(x-y) \end{bmatrix}$$

$$[Df\circ g](1,1) = \begin{bmatrix} e^0\sec^2(e^0-1) - e^0 & -e^0\sec^2(e^0-1) + e^0 \\ 2e^0 - 2(0) & -2e^0 + 2(0) \end{bmatrix}$$

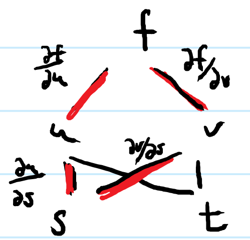
$$= \boxed{\begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}}$$

8 Similar to **7**. The point is that it would be easier to use the chain rule.

9 $f(u,v) = \cos(u)\sin(v)$

$T(s,t) = (\underset{u}{\cos(t^2 s)}, \underset{v}{\log\sqrt{1+s^2}})$

Using the shortcut method, we can make the tree:



Then $\frac{\partial f}{\partial s} = \left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial u}{\partial s}\right) + \left(\frac{\partial f}{\partial v}\right)\left(\frac{\partial v}{\partial s}\right)$

- $\frac{\partial f}{\partial u} = -\sin(u)\sin(v)$
- $\frac{\partial u}{\partial s} = -t^2\sin(t^2 s)$
- $\frac{\partial f}{\partial v} = \cos(u)\cos(v)$
- $\frac{\partial v}{\partial s} = \frac{s}{1+s^2}$

When $(s,t) = (1,0)$, $(u,v) = (\cos(0), \log\sqrt{2})$
$= (1, \log\sqrt{2})$

- $\frac{\partial f}{\partial u}(1,\log\sqrt{2}) = -\sin(1)\sin(\log\sqrt{2})$
- $\frac{\partial f}{\partial v}(1,\log\sqrt{2}) = \cos(1)\cos(\log\sqrt{2})$
- $\frac{\partial u}{\partial s}(1,0) = 0$
- $\frac{\partial v}{\partial s}(1,0) = \frac{1}{2}$.

So $\frac{\partial(f\circ T)}{\partial s}(1,0) = (-\sin(1)\sin(\log\sqrt{2})\cdot 0 + \cos(1)\cos(\log\sqrt{2})\cdot\frac{1}{2}$
$= \boxed{\frac{1}{2}\cos(1)\cos(\log\sqrt{2})}$.

10 $T'(t) = (\nabla T)(\vec{\sigma}(t)) \cdot \vec{\sigma}\,'(t)$
$= (2x, 2y, 2z)\big|_{\vec{\sigma}(t)} \cdot (-\sin(t), \cos(t), 1)$

$= (2\cos(t), 2\sin(t), 2t)\cdot(-\sin(t), \cos(t), 1)$
$= \boxed{2t}$

Using linear approximation at time $t_0 = \frac{\pi}{2}$,

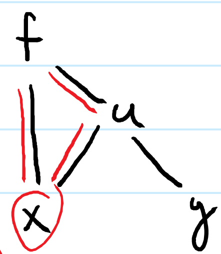
we get the temp at time $\frac{\pi}{2} + 0.01 \approx$

$$T(\vec{\sigma}(\tfrac{\pi}{2})) + (T\circ\vec{\sigma})'(\tfrac{\pi}{2})\left((\tfrac{\pi}{2}+0.01) - \tfrac{\pi}{2}\right)$$
$$= \boxed{\left(1+\frac{\pi^2}{4}\right) + \pi(0.01)}$$

$\boxed{12}$ See notes from 2/8

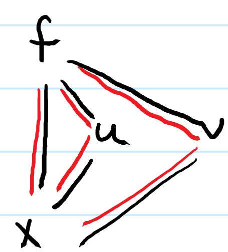
$\boxed{13}$ See notes from 2/8

$\boxed{17a}$ $h = f(x, u(x,y))$. Using the tree:



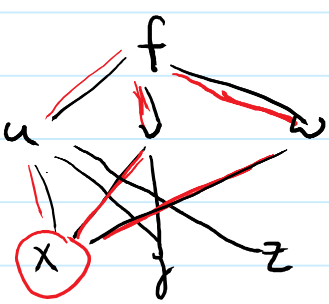
$$\boxed{\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x} + \left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial u}{\partial x}\right)}$$

$\boxed{17b}$ $h = f(x, u(x), v(x))$. Using the tree:



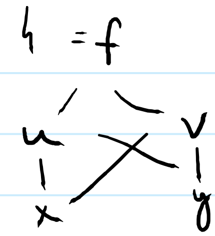
$$\boxed{\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x} + \left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial f}{\partial v}\right)\left(\frac{\partial v}{\partial x}\right)}$$

$\boxed{17c}$ $h = f(u(x,y,z), v(x,y), w(x))$



$$\boxed{\frac{\partial h}{\partial x} = \left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial f}{\partial v}\right)\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial f}{\partial w}\right)\left(\frac{dw}{dx}\right)}$$

18 Using the tree:



$$\boxed{\frac{\partial h}{\partial x} = \left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial f}{\partial v}\right)\left(\frac{\partial v}{\partial x}\right)}$$

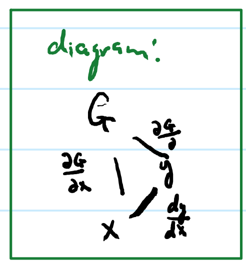
- $\frac{\partial f}{\partial u} = \frac{(u^2-v^2)(2u)-(u^2+v^2)2u}{(u^2-v^2)^2} = \frac{-4uv^2}{(u^2-v^2)^2}$
- $\frac{\partial f}{\partial v} = \frac{2v(u^2-v^2) + 2v(u^2+v^2)}{(u^2-v^2)^2} = \frac{4vu^2}{(u^2-v^2)^2}$
- $\frac{\partial u}{\partial x} = -e^{-x-y}$
- $\frac{\partial v}{\partial x} = ye^{xy}$

$$\boxed{\frac{\partial h}{\partial x} = \left(\frac{-4uv^2}{(u^2-v^2)^2}\right)\left(-e^{-x-y}\right) + \left(\frac{4vu^2}{(u^2-v^2)^2}\right)\left(ye^{xy}\right)}$$

You should write, u, v in terms of x, y but I'm okay with this.

19 a If $G(x, y(x)) = 0$, then

$$\frac{d}{dx}(G(x, y(x))) = \frac{d}{dx}(0) = 0$$



Chain rule $\Rightarrow \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\cdot\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{\left(-\frac{\partial G}{\partial x}\right)}{\left(\frac{\partial G}{\partial y}\right)} \quad \text{as long as } \frac{\partial G}{\partial y} \neq 0$$

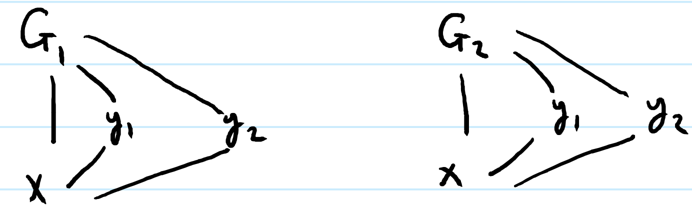
Rmk: This is the proof of implicit differentiation from Calc I.

19b Suppose

$$G_1(x, y_1(x), y_2(x)) = 0$$
$$G_2(x, y_1(x), y_2(x)) = 0$$

differentiate both eqns w.r.t. x using chain rule:

Diagrams:



$$\frac{\partial}{\partial x}\left(G_1(x, y_1(x), y_2(x))\right) = \frac{\partial}{\partial x}(0)$$

$$\Rightarrow \quad \frac{\partial G_1}{\partial x} + \left(\frac{\partial G_1}{\partial y_1}\right)\left(\frac{dy_1}{dx}\right) + \left(\frac{\partial G_1}{\partial y_2}\right)\left(\frac{dy_2}{dx}\right) = 0$$

Similarly,

$$\frac{\partial}{\partial x}\left(G_2(x, y_1(x), y_2(x))\right) = \frac{\partial}{\partial x}(0)$$

$$\Rightarrow \quad \frac{\partial G_2}{\partial x} + \left(\frac{\partial G_2}{\partial y_1}\right)\left(\frac{dy_1}{dx}\right) + \left(\frac{\partial G_2}{\partial y_2}\right)\left(\frac{dy_2}{dx}\right) = 0$$

Then

$$\left(\frac{\partial G_1}{\partial y_1}\right)\left(\frac{dy_1}{dx}\right) + \left(\frac{\partial G_1}{\partial y_2}\right)\left(\frac{dy_2}{dx}\right) = -\frac{\partial G_1}{\partial x}$$

$$\left(\frac{\partial G_2}{\partial y_1}\right)\left(\frac{dy_1}{dx}\right) + \left(\frac{\partial G_2}{\partial y_2}\right)\left(\frac{dy_2}{dx}\right) = -\frac{\partial G_2}{\partial x}$$

Solving this system of eqns gives:

$$\frac{dy_1}{dx} = \frac{-\left(\frac{\partial G_2}{\partial y_2}\right)\cdot\left(\frac{\partial G_1}{\partial x}\right) + \left(\frac{\partial G_1}{\partial y_2}\right)\cdot\left(\frac{\partial G_2}{\partial x}\right)}{\left(\frac{\partial G_1}{\partial y_1}\right)\left(\frac{\partial G_2}{\partial y_2}\right) - \left(\frac{\partial G_1}{\partial y_2}\right)\left(\frac{\partial G_2}{\partial y_1}\right)} \quad \text{and}$$

$$\frac{dy_2}{dx} = \frac{\left(\frac{\partial G_2}{\partial y_1}\right)\cdot\left(\frac{\partial G_1}{\partial x}\right) - \left(\frac{\partial G_1}{\partial y_1}\right)\cdot\left(\frac{\partial G_2}{\partial x}\right)}{\left(\frac{\partial G_1}{\partial y_1}\right)\left(\frac{\partial G_2}{\partial y_2}\right) - \left(\frac{\partial G_1}{\partial y_2}\right)\left(\frac{\partial G_2}{\partial y_1}\right)}$$

19c Let $x^2 + y^3 + e^y = 0$ and let $G(x,y) = x^2 + y^3 + e^y$.

By part (a), $\boxed{\frac{dy}{dx} = \frac{\left(-\frac{\partial G}{\partial x}\right)}{\left(\frac{\partial G}{\partial y}\right)} = \frac{-2x}{3y^2 + e^y}}$

---

34 Let $f: \mathbb{R}^n \to \mathbb{R}^m$, $g: \mathbb{R}^p \to \mathbb{R}^q$

(a) For $(f \circ g)(x) = f(g(x))$ to make sense, the output of g must be in the domain of f. so $q = m$

(b) For $(g \circ f)(x) = g(f(x))$ to make sense, the output of f must be in the domain of g, so $n = p$

(c) With the reasoning as before, $f \circ f$ makes sense $\Leftrightarrow$ $m=n$.

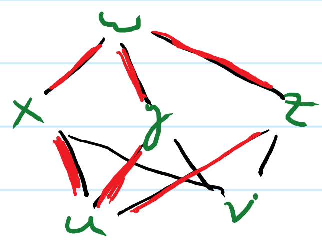
3c If $w = x^2+y^2+z^2$, $x = uv$, $y = u\cos v$, $z = u\sin v$.

Then $$\frac{\partial w}{\partial u} = \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial y}{\partial u}\right) + \left(\frac{\partial w}{\partial z}\right)\left(\frac{\partial z}{\partial u}\right)$$

$$= (2x)(v) + (2y)(\cos v) + (2z)(\sin v)$$
$$= (2uv)(v) + (2u\cos v)(\cos v) + (2u\sin v)(\sin v)$$

So $$\left(\frac{\partial w}{\partial u}\right)(1,0) = 0 + 2 + 0$$
$$= \boxed{2}$$

Diagram!



§2.6: 1, 2, 4, 17, 18, 22, 26, 32:

1a Since $\vec{u} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$ is a unit vector,

$$(D_{\vec{u}} f)(1,1,2) = (\nabla f)(1,1,2) \cdot \vec{u}.$$

Calculate:

$\nabla f = (z^2, 3y^2, 2zx)$.

$(\nabla f)(1,1,2) = (4, 3, 4)$.

So $(D_{\vec{u}} f)(1,1,2) = (4,3,4) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right) = \boxed{\frac{10}{\sqrt{5}}}$

---

2 Note that the direction vectors for each problem is unit, so we use $(D_{\vec{u}} f) = (\nabla f) \cdot \vec{u}$.

(a) $\nabla f = (1 + 2y, 2x - 6y)$

$(\nabla f)(1,2) = (5, -10)$

$(D_{\vec{v}} f)(1,2) = (5, -10) \cdot \left(\frac{3}{5}, \frac{4}{5}\right)$

$= 3 - 8$

$= \boxed{-5}$

(b) $\nabla f = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$

$\nabla f(1,0) = (1, 0)$

$(D_{\vec{v}} f)(1,0) = (1,0) \cdot \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

$= \boxed{\frac{2}{\sqrt{5}}}$

(c) $\nabla f = (e^x \cos(\pi y), -\pi e^x \sin(\pi y))$

$(\nabla f)(0,-1) = (-1, 0)$

$(D_{\vec{v}} f)(0,-1) = (-1,0) \cdot \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}$

d) $(\nabla f) = (y^2 + 3x^2 y,\ 2xy + x^3)$

$(\nabla f)(4,-2) = (-92,\ 48)$

$(D_v f)(4,-2) = (-92, 48) \cdot \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$

$= \frac{-92}{\sqrt{10}} + \frac{144}{\sqrt{10}} = \boxed{\frac{52}{\sqrt{10}}}$

4) We want a vector $\vec{u}$ such that $(D_{\vec{u}} f)(2,1) = 0$.
So we need $\vec{u}$ so that $(\nabla f)(2,1) \cdot \vec{u} = 0$

$\nabla f = (-\pi y \sin(\pi x) - \cos(\pi y),\ \cos(\pi x) + \pi x \sin(\pi y))$

$(\nabla f)(2,1) = (-\pi \sin(2\pi) - \cos(\pi),\ \cos(2\pi) + 2\pi \sin(\pi))$

$= (1, 1)$.

If we want $(D_{\vec{u}} f)(2,1) = (\nabla f)(2,1) \cdot (u_1, u_2) = 0$

$\Rightarrow (1,1) \cdot (u_1, u_2) = 0$

$\Rightarrow u_1 + u_2 = 0$.

So we can take $\boxed{\vec{u} = \frac{1}{\sqrt{2}}(1,-1)}$

17) This is essentially a chain rule question.
Use $(f \circ g)'(1) = (\nabla f)(g(1)) \cdot g'(1)$.

18) Similar to problem 2, but in 3-dimensions.

22) a) The direction of steepest increase is ∇T, so the direction of steepest decrease is

$-\nabla T = -(-2x e^{-x^2-2y^2-3z^2},\ -4y e^{-x^2-2y^2-3z^2},\ -6z e^{-x^2-2y^2-3z^2})$.

$(-\nabla T)(1,1,1) = (2e^{-6},\ 4e^{-6},\ 6e^{-6})$.

b) The rate of change in temperature as the boat travels e^8 m/s in the $-\nabla T(1,1,1)$ direction is

$(D_{\vec{u}} f)(1,1,1)$ where $\vec{u} = e^8 \left(\frac{-\nabla T(1,1,1)}{\|-\nabla T(1,1,1)\|} \right)$

(c) The set of possible directions is any unit vector $\vec{u}$ satisfying

$$(D_{e^{?}\vec{u}} f)(1,1,1) \leq \sqrt{14}\, e^{2}.$$

26 The direction of steepest increase is

$$(\nabla z)(1,1) = (-2a, -2b)$$

The ball will roll in the $-(\nabla z)(1,1) = (2a, 2b)$ direction.

Column vectors

32 We view (Df) as a fcn from $\mathbb{R}^3 \to \mathbb{R}$ by $[(Df)(x,y,z)]\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

matrix mult.

If $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ gets sent to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ under Df, then

$$[Df]\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\Leftrightarrow a\left(\frac{\partial f}{\partial x}\right) + b\left(\frac{\partial f}{\partial y}\right) + c\left(\frac{\partial f}{\partial z}\right) = 0$$

$$\Leftrightarrow (a,b,c) \cdot \nabla f = 0$$

$$\Leftrightarrow (a,b,c) \perp \nabla f$$

which is what we wanted.