§2.6: 8, 15, 20, 25:

For these problems, we frequently make use of the fact that the gradient of a fcn is $\perp$ to level sets

8 Find tangent planes to the surfaces at the given points:

(a) $x^2+2y^2+3xz=10$ at $(1,2,\frac{1}{3})$.

Let $F(x,y,z)=x^2+2y^2+3xz$. Then the surface

$x^2+2y^2+3xz=10$ is just the level surface

$F(x,y,z)=10$.

Therefore, $\nabla F(1,2,\frac{1}{3})$ is $\perp$ to the surface at $(1,2,\frac{1}{3})$

So we can use $\vec{N}=\nabla F(1,2,\frac{1}{3})$ as the normal vector to the tangent plane.

$$\nabla F(1,2,\tfrac{1}{3})=(2x+3z,\ 4y,\ 3x)\Big|_{(1,2,\frac{1}{3})}=(3,8,3).$$

Clearly the point $P=(1,2,\frac{1}{3})$ is on the plane.

So we can use the eqn: $3x+8y+3z=(3,8,3)\cdot(1,2,\frac{1}{3})$

$= \boxed{3x+8y+3z=20}$

(8b) $y^2-x^2=3$ at $(1,2,8)$

Using part (a) as a model,

Let $F(x,y,z)=y^2-x^2$.

Then we take $\vec{N} = \nabla F(1,2,8) = (-2x, 2y, 0)\Big|_{(1,2,8)} = (-2,4,0)$

$P_0 = (1,2,8)$

So the eqn is:

$$-2x + 4y + 0z = (-2,4,0)\cdot(1,2,8)$$

$$\Rightarrow \quad \boxed{-2x + 4y = 6}$$

8c) $xyz = 1$ at $(1,1,1)$

Let $F(x,y,z) = xyz$.

Then $\vec{N} = \nabla F(1,1,1) = (yz, xz, xy)\Big|_{(1,1,1)} = (1,1,1)$.

$P_0 = (1,1,1)$

So the eqn is: $x + y + z = (1,1,1)\cdot(1,1,1)$

$$\Rightarrow \quad \boxed{x + y + z = 3}$$

15) Goal: Using the technique in problem 8. Show the eqn of the plane tangent to the graph of $z = f(x,y)$ at (a,b) is

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

(which is the formula we already knew.).

pf: If $f(x,y) = z$, then $f(x,y) - z = 0$. Define the fcn

$$F(x,y,z) = f(x,y) - z.$$

Then the graph of $f(x,y)$ is the level surface

$F(x,y,z) = 0$. We know $\nabla F(a,b,f(a,b))$ is $\perp$ to the tan. plane of $F(x,y,z) = 0$ at $(a,b,f(a,b))$. and we know $P_0 = (a,b,f(a,b))$ is a point on the tan plane.

Now $\nabla F(a,b,f(a,b)) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)\Big|_{(a,b,f(a,b))}$

$$= \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b), -1\right)$$

So the tan. plane is given by:

$$\left(\frac{\partial f}{\partial x}\right)(a,b)x + \frac{\partial f}{\partial y}(a,b)y - z = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b), -1\right)\cdot(a,b,f(a,b))$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)(a,b)x + \frac{\partial f}{\partial y}(a,b)y - z = \frac{\partial f}{\partial x}(a,b)\cdot a + \frac{\partial f}{\partial y}(a,b)\cdot b - f(a,b)$$

$$\Rightarrow z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

Solve for z

$\boxed{20}$ Let $F(x,y,z) = x^2 + 4y^2 - z^2$. Then for any point (a,b,c) on the hyperboloid, $\nabla F(a,b,c)$ is $\perp$ to $x^2+4y^2-z^2=4$ at (a,b,c).

Since we want the tan. plane to be $\parallel$ to $2x+2y+z=5$, we want a point (a,b,c) where $\nabla F(a,b,c) = \lambda(2,2,1)$ and $a^2+4b^2-c^2=4$ for some scalar λ.

The second eqn is there to ensure (a,b,c) actually lies on the surface.

Now $\nabla F = (2x, 8y, -2z)$ so we want a pt. (a,b,c) with

$$(2a, 8b, -2c) = \lambda(2,2,1),$$
$$a^2+4b^2-c^2=4$$

or

$$\begin{aligned} &\text{I} \quad 2a = \lambda 2 \\ &\text{II} \quad 8b = \lambda 2 \\ &\text{III} \quad -2c = \lambda \\ &\text{IV} \quad a^2 + 4b^2 - c^2 = 4. \end{aligned} \Bigg\} \Rightarrow \quad \lambda = \boxed{a = 4b = -2c} \text{ or } \boxed{a = 4b, \ c = -2b}$$

Thus, IV $\Rightarrow (4b)^2 + 4b^2 - (-2b)^2 = 4$

$\Rightarrow 16b^2 + 4b^2 - 4b^2 = 4$

$\Rightarrow \boxed{b = \pm \frac{1}{2}.}$

If $b = \frac{1}{2}$, $a = 2$, $c = -1 \Rightarrow (a, b, c) = (2, \frac{1}{2}, -1)$

if $b = -\frac{1}{2}$, $a = -2$, $c = 1 \Rightarrow (a, b, c) = (-2, -\frac{1}{2}, 1)$

25 Suppose $f: \mathbb{R}^n \to \mathbb{R}$ satisfies

$f(\vec{x}) = f(-\vec{x})$, and is diff'ble.

If we compute the total derivative on both sides:

$$[Df](\vec{x}) = [\overrightarrow{Df}](-\vec{x})$$

We can pull the $(-)$ sign out of $(Df)(-\vec{x})$ to get

$$\begin{aligned} & [Df](\vec{x}) = -[Df](\vec{x}) \\ \Rightarrow \quad & [Df](\vec{x}) + [Df](\vec{x}) = \vec{0} \\ \Rightarrow \quad & 2[Df](\vec{x}) = \vec{0} \\ \Rightarrow \quad & \boxed{[Df](\vec{x}) = \vec{0}} \end{aligned}$$

§3.1: 1, 2, 3, 9, 10, 23, 32:

1,2,3 Are simple (but tedious) calculations. Here is 3 for example.

$f(x,y) = \cos(xy^2)$.

$\frac{\partial f}{\partial x} = -y^2 \sin(xy^2)$ $\qquad$ $\frac{\partial f}{\partial y} = -2xy \sin(xy^2)$

$\frac{\partial^2 f}{\partial x^2} = -y^2 \cos(xy^2)$ $\qquad$ $\frac{\partial^2 f}{\partial y^2} = -2x \sin(xy^2) - 4x^2y^2 \cos(xy^2)$

$$\underbrace{\frac{\partial^2 f}{\partial y \partial x} = -2y \sin(xy^2) - 2xy^3\cos(xy^2) \qquad \frac{\partial^2 f}{\partial x \partial y} = -2y \sin(xy^2) - 4xy^3 \sin(xy^2)}_{\text{equal!}}$$

9 If there were a C^2 fcn with

$$f_x = 2x - 5y, \quad f_y = 4x + y,$$

then $f_{xy} = -5$ while $f_{yx} = 4$.

This contradicts Clairaut's Thm.

So $\boxed{\text{no}}$

10 $u_t = -k e^{-kt} \sin(x)$.

$u_x = e^{-kt} \cos(x)$.

$u_{xx} = -e^{-kt} \sin(x)$.

$k u_{xx} = -k e^{-kt} \sin(x) = u_t$.

yes, it is a sol'n.

23 let $f(x,y): \mathbb{R}^2 \to \mathbb{R}$, then $\frac{d}{dt}(f \circ \vec{c})(t) = (\nabla f)(\vec{c}(t)) \cdot \vec{c}\,'(t)$

We view ∇f as a fcn $\mathbb{R}^2 \to \mathbb{R}^2$, $\nabla f(x,y) = \langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \rangle$.

Using the chain and product rules:

$$\frac{d^2}{dt^2}(f\circ\vec{c})(t) = \frac{d}{dt}\left(\nabla f(\vec{c}(t))\cdot\vec{c}\,'(t)\right)$$
$$= \frac{d}{dt}\left(\nabla f(\vec{c}(t))\right)\cdot\vec{c}\,'(t) + \nabla f(\vec{c}(t))\cdot\vec{c}\,''(t)$$
$$= \left(\underbrace{\left.\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y\partial x} \\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2}\end{bmatrix}\right|_{\vec{c}(t)} \begin{bmatrix} x'(t) \\ y'(t)\end{bmatrix}}_{\text{This is a vector}}\right)\cdot\vec{c}\,'(t) + \nabla f(\vec{c}(t))\cdot\vec{c}\,''(t)$$

You can reduce if you like, but this is fine.

$\boxed{32}$ (a) For $(x,y)\neq(0,0)$

$$\frac{\partial f}{\partial x} = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$$
$$\frac{\partial f}{\partial y} = \frac{-x(y^4+4x^2y^2-x^4)}{(x^2+y^2)^2}$$

(b) $$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}(0-0) = \boxed{0}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \frac{1}{h}(0-0)$$
$$= 0.$$

(c) $$\frac{\partial^2 f}{\partial x\partial y}(0,0) = \lim_{h\to 0}\frac{\frac{\partial f}{\partial y}(h,0)-\frac{\partial f}{\partial y}(0,0)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\left(\underbrace{\frac{(-h)(-h^4)}{h^4}}_{\frac{\partial f}{\partial y}(h,0)} - 0\right)$$
$$= \lim_{h\to 0} 1 = 1$$

$\frac{\partial^2 f}{\partial y \partial x}(0,0) = -1$ using a similar calculation.

(d) Equality of mixed partials fails because the 2^{nd} order partials are not cont. at $(0,0)$.

§3.3: 1, 5, 7, 8, 9, 13, 20, 22, 28:

1 - 13 are similar, we will do 1 & 9.

(1) $f(x,y) = x^2 - y^2 + xy$.

$$\frac{\partial f}{\partial x} = 2x + y$$
$$\frac{\partial f}{\partial y} = -2y + x$$

$$\frac{\partial^2 f}{\partial x^2} = 2$$
$$\frac{\partial^2 f}{\partial y \partial x} = 1$$
$$\frac{\partial^2 f}{\partial y^2} = -2.$$

Critical points when $\left.\begin{array}{l} 2x+y=0 \\ -2y+x=0 \end{array}\right\} \Rightarrow \boxed{y=-2x}$

$\Rightarrow 4x + x = 0 \Rightarrow x = 0 \Rightarrow y = 0$

So only $\boxed{(0,0)}$

Since $D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial y \partial x}\right)^2$
$= -4 - 1 < 0$

$(0,0)$ is a saddle pt.

(9) $f(x,y) = \cos(x^2+y^2)$. $\quad \frac{\partial f}{\partial x} = -2x \sin(x^2+y^2)$, $\quad \frac{\partial f}{\partial y} = -2y \sin(x^2+y^2)$

$\frac{\partial^2 f}{\partial x^2} = -2\sin(x^2+y^2) - 4x^2\cos(x^2+y^2)$, $\quad \frac{\partial^2 f}{\partial y^2} = -2\sin(x^2+y^2) - 4y^2\cos(x^2+y^2)$

$$\frac{\partial^2 f}{\partial y \partial x} = -4xy\cos(x^2+y^2)$$

For $\left(\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right)$:

$$\frac{\partial^2 f}{\partial x^2}\left(\sqrt{\tfrac{\pi}{2}}, \sqrt{\tfrac{\pi}{2}}\right) = -2\sin(\pi) - 2\pi\cos(\pi)$$
$$= 2\pi$$

$$\frac{\partial^2 f}{\partial y^2}\left(\sqrt{\tfrac{\pi}{2}}, \sqrt{\tfrac{\pi}{2}}\right) = 2\pi$$

$$\frac{\partial^2 f}{\partial y \partial x}\left(\sqrt{\tfrac{\pi}{2}}, \sqrt{\tfrac{\pi}{2}}\right) = -2\pi\cos(\pi)$$
$$= 2\pi$$

$$\Rightarrow D\left(\sqrt{\tfrac{\pi}{2}}, \sqrt{\tfrac{\pi}{2}}\right) = (2\pi)(2\pi) - (2\pi)^2 = 0.$$

The 2^{nd} der. test fails! We get no info.

On the other hand, $f\left(\sqrt{\tfrac{\pi}{2}}, \sqrt{\tfrac{\pi}{2}}\right) = \cos(\pi) = -1.$

Since we know $-1 \le \cos\theta \le 1$ for any θ,

This must be a local min!

With a similar argument, $f(0,0)$ is a max.

$f(0, \sqrt{\pi})$ is a min.

20 (a) $D(4,2) = (1)(5) - (3)^2 < 0 \Rightarrow$ Saddle.

(b) $D(4,2) = (2)(4) - (-1)^2 > 0,$ $f_{xx}(4,2) = 2 > 0$ $\Big\} \Rightarrow$ local min.

(c) $D(4,2) = (-2)(3) - (1)^2 < 0 \Rightarrow$ Saddle

22 See what happens at $(0,0)$ as k changes.

$\frac{\partial f}{\partial x} = 2x + ky$ $\qquad \frac{\partial^2 f}{\partial x^2} = 2$ $\qquad \frac{\partial^2 f}{\partial y^2} = 2$

$\frac{\partial f}{\partial y} = 2y + kx$ $\qquad \frac{\partial^2 f}{\partial y \partial x} = k$

So $D(x,y) = 4 - k^2$. D is constant, and switches sign at $\boxed{k = \pm 2}$

So the graph changes at $k = \pm 2$.

(28) If (x,y,z) is on the plane, then $z = 20 - x + \frac{y}{2}$.

So any point on the plane looks like $(x, y, 20 - x + \frac{y}{2})$.
The distance from the origin is then

$$\sqrt{x^2 + y^2 + (20 - x + \tfrac{y}{2})^2}.$$

we will minimize the distance squared:

$$d(x,y) = x^2 + y^2 + (20 - x + \tfrac{y}{2})^2.$$

$$\nabla d = (2x - 2(20 - x + \tfrac{y}{2}), 2y + 20 - x + \tfrac{y}{2})$$

$$= (4x - y - 40, \tfrac{5}{2}y - x + 20)$$

To find crit. pts, we need:

$$4x - y - 40 = 0$$
$$\tfrac{5}{2}y - x + 20 = 0$$

$$\Rightarrow \quad 4x - y = 40$$
$$2x - 5y = 40$$

$$\Rightarrow \boxed{(x,y) = \left(\tfrac{80}{9}, -\tfrac{40}{9}\right)}$$