§ 4.1: 5, 6, 7, 8, 12, 13, 15, 19, 20:

For 5-8: $\vec{c}_1(t) = (e^t, \sin t, t^3) \Rightarrow \vec{c}_1{}'(t) = (e^t, \cos t, 3t^2)$

$\vec{c}_2(t) = (e^{-t}, \cos t, -2t^3) \Rightarrow \vec{c}_2{}'(t) = (-e^{-t}, -\sin t, -6t^2)$

5. 1st way: $\vec{c}_1(t) + \vec{c}_2(t) = (e^t + e^{-t}, \sin t + \cos t, -t^3)$

$\frac{d}{dt}(\vec{c}_1(t) + \vec{c}_2(t)) = \boxed{(e^t - e^{-t}, \cos t - \sin t, -3t^2)}$.

2nd way (using rules): $\frac{d}{dt}(\vec{c}_1(t) + \vec{c}_2(t)) = \vec{c}_1{}'(t) + \vec{c}_2{}'(t)$

$= (e^t, \cos t, 3t^2) + (-e^{-t}, -\sin t, -6t^2)$

$= \boxed{(e^t - e^{-t}, \cos t - \sin t, -3t^2)}$ ✓.

6. 1st way: $\vec{c}_1(t) \cdot \vec{c}_2(t) = (e^t)(e^{-t}) + (\cos t)(\sin t) - 2t^6$

$= 1 + \cos(t)\sin(t) - 2t^6$

$\Rightarrow \frac{d}{dt}(\vec{c}_1(t) \cdot \vec{c}_2(t)) = \boxed{-12t^5 - \sin^2 t + \cos^2 t}$.

2nd way (using rules):

$\frac{d}{dt}(\vec{c}_1(t) \cdot \vec{c}_2(t)) = (\vec{c}_1{}'(t) \cdot \vec{c}_2(t)) + (\vec{c}_1(t) \cdot \vec{c}_2{}'(t))$

$= ((e^t, \cos t, 3t^2) \cdot (e^{-t}, \cos t, -2t^3)) + ((e^t, \sin t, t^3) \cdot (-e^{-t}, -\sin t, -6t^2))$

$= 1 + \cos^2 t - 6t^5 + (-1 - \sin^2(t) - 6t^5)$

$= \boxed{-12t^5 - \sin^2 t + \cos^2 t}$ ✓

7. 1st way: $\vec{c}_1(t) \times \vec{c}_2(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ e^t & \sin t & t^3 \\ e^{-t} & \cos t & -2t^3 \end{vmatrix}$

$= (-2t^3 \sin t - t^3 \cos t, t^3 e^{-t} + 2t^3 e^t, e^t \cos t - e^{-t} \sin t)$

$= (t^3(-2\sin t - \cos t), t^3(e^{-t} + 2e^t), e^t \cos t - e^{-t} \sin t)$

$\frac{d}{dt}(\vec{c}_1(t) \times \vec{c}_2(t)) = (3t^2(-2\sin t - \cos t) + t^3(-2\cos t + \sin t))\vec{i} + (3t^2(e^{-t} + 2e^t) + t^3(-e^{-t} + 2e^t))\vec{j}$

$+ (\cos t(e^t - e^{-t}) + \sin t(e^{-t} - e^t))\vec{k}$

2^{nd} way (using rules): $\frac{d}{dt}(\vec{c}_1(t)\times\vec{c}_2(t)) = (\vec{c}_1'(t)\times\vec{c}_2(t)) + (\vec{c}_1(t)\times c_2'(t))$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ e^t & \cos t & 3t^2 \\ e^{-t} & \cos t & -2t^3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ e^t & \sin t & t^3 \\ -e^{-t} & -\sin t & -6t^2 \end{vmatrix}$$

$$= (-2t^3\cos t - 3t^2\cos t,\ 3t^2e^{-t} + 2t^3e^t,\ e^t\cos t - e^{-t}\cos t) + (-6t^2\sin t + t^3\sin t,\ -t^3e^{-t} + 6t^2e^t,\ -e^t\sin(t) + e^{-t}\sin(t))$$

(they're the same.)

8 $\underline{1^{st}\text{ way}}$: $\{\vec{c}_1(t)\cdot[2\vec{c}_2(t) + \vec{c}_1(t)]\} = (e^t, \sin t, t^3)\cdot(2e^{-t}+e^t, 2\cos t + \sin t, -3t^3)$

$$= (e^t)(2e^{-t}+e^t) + (\sin t)(2\cos t + \sin t) - 3t^6$$

$$= 2 + e^{2t} + 2\sin t\cos t + \sin^2 t - 3t^6$$

$$\frac{d}{dt}\{\vec{c}_1(t)\cdot[2\vec{c}_2(t)+\vec{c}_1(t)]\} = \boxed{2e^{2t} + 2\cos^2 t - 2\sin^2 t + 2\sin t\cos t - 18t^5.}$$

$\underline{2^{nd}\text{ way}}$: $\vec{c}_1(t)\cdot[2\vec{c}_2(t)+\vec{c}_1(t)]$

$$\frac{d}{dt}(\vec{c}_1\cdot[2\vec{c}_2(t) + \vec{c}_1(t)]) = (\vec{c}_1'(t)\cdot(2\vec{c}_2(t)+\vec{c}_1(t))) + (\vec{c}_1(t)\cdot(2\vec{c}_2'(t)+\vec{c}_1'(t)))$$

$$= ((e^t, \cos t, 3t^2)\cdot(2e^{-t}+e^t, 2\cos t + \sin t, -3t^3)) + ((e^t, \sin t, t^3)\cdot(-2e^{-t}+e^t, -2\sin t + \cos t, -9t^2))$$

$$= (\cancel{2} + e^{2t} + 2\cos^2 t + \cos t\sin t - 9t^5) + (\cancel{-2} + e^{2t} - 2\sin^2 t + \sin t\cos t - 9t^5)$$

$$= \boxed{2e^{2t} + 2\cos^2 t - 2\sin^2 t + 2\sin t\cos t - 18t^5} \quad \checkmark$$

12 $\vec{c}(0) = (3,4,0)$, $\vec{v}(0) = (1,1,-2)$ $\vec{a}(t) = (0,0,6)$

$\vec{v}'(t) = (0,0,6) \Rightarrow \vec{v}(t) = (0,0,6t) + \vec{u}$, $\vec{u}$ a constant vector.

plug in $t=0$.

$\vec{v}(0)=(1,1,-2) \Rightarrow (1,1,-2) = \underbrace{(0,0,0)}_{\text{plug in } t=0} + \vec{u}_1$

$\Rightarrow \vec{u}_1 = (1,1,-2)$

$\Rightarrow \boxed{\vec{v}(t) = (1,1,6t-2).}$

Now $\vec{c}\,'(t) = (1,1,6t-2)$

integrate $\Rightarrow \vec{c}(t) = (t,t,3t^2-2t) + \vec{u}_2$, $\vec{u}_2$ a constant vector.

$\vec{c}(0) = (3,4,0)$ plug in 0 in $\vec{c}(t)$.

$\Rightarrow (3,4,0) = \underbrace{(0,0,3(0)^2-2(0))}_{} + \vec{u}_2$

$\vec{u}_2 = (3,4,0)$

$\Rightarrow \boxed{\vec{c}(t) = (t+3, t+4, 3t^2-2t)}$

$\boxed{13}$ $\vec{a}(t) = (2,-6,-4)$, $\vec{v}(0) = (-5,1,3)$, $\vec{r}(0) = (6,-2,1)$

$\vec{a}(t) = (2,-6,-4) \Rightarrow \vec{v}(t) = (2t,-6t,-4t) + \vec{u}_1$, $\vec{u}_1$ const.

$\vec{v}(0) = (-5,1,3)$

$\Rightarrow (-5,1,3) = (2(0),-6(0),-4(0)) + \vec{u}_1$

$\Rightarrow \vec{u}_1 = (-5,1,3)$

$\Rightarrow \vec{v}(t) = (2t-5, -6t+1, -4t+3).$

Now $\vec{r}(t) = (t^2-5t, -3t^2+t, -2t^2+3t) + \vec{u}_2$, $\vec{u}_2$ const.

$\vec{r}(0) = (6,-2,1) \Rightarrow$

$(6,-2,1) = (0,0,0) + \vec{u}_2$

$\Rightarrow \vec{u}_2 = (6,-2,1)$

$\Rightarrow \vec{r}(t) = (t^2-5t+6, -3t^2+t-2, -2t^2+3t+1).$

$\vec{r}(t)$ intersects yz-plane when $x=0 \Rightarrow t^2-5t+6=0$

$\Rightarrow (t-3)(t-2)=0$

$\Rightarrow t=3$ or 2.

So the two points are: $\vec{r}(2) = (0,-12,-1)$

$\vec{r}(3) = (0,-26,-8)$

15 $\vec{r}(t) = \langle 6t, 3t^2, t^3 \rangle$

$\vec{v}(t) = \langle 6, 6t, 3t^2 \rangle$

$\vec{a}(t) = \langle 0, 6, 6t \rangle.$

$\vec{a}(0) = \langle 0, 6, 0 \rangle.$

$\Rightarrow$ Force at $t=0$ is $\boxed{\vec{F} = m\vec{a}(0) = \langle 0, 6m, 0 \rangle}$

19 Assume $\vec{c}\,''(t) \perp \vec{c}\,'(t)$ for all t.

Note: $\|\vec{c}\,'(t)\|$ is constant $\Leftrightarrow$ $\|\vec{c}\,'(t)\|^2$ is constant.

speed.

We calculate $\frac{d}{dt}(s(t)^2) = \frac{d}{dt}(\|\vec{c}\,'(t)\|^2)$

$$= \frac{d}{dt}(\vec{c}\,'(t) \cdot \vec{c}\,'(t))$$

$$= (\vec{c}\,''(t) \cdot \vec{c}\,'(t)) + (\vec{c}\,'(t) \cdot \vec{c}\,''(t))$$

$$= 2(\vec{c}\,''(t) \cdot \vec{c}\,'(t)).$$

Since $\vec{c}\,''(t) \perp \vec{c}\,'(t)$ for all t,

$$\frac{d}{dt}(\|\vec{c}\,'(t)\|^2) = 2(\vec{c}\,''(t) \cdot \vec{c}\,'(t))$$
$$= 0.$$

Now Since the derivative of $\|\vec{c}\,'(t)\|^2 = 0$,

$\|\vec{c}\,'(t)\|^2$ is constant $\Rightarrow$ $s(t) = \|\vec{c}\,'(t)\|$ is constant.

20 The fcn $\|\vec{r}(t)\|$ is a function from $\mathbb{R}$ to $\mathbb{R}$.

Therefore, $\|\vec{r}(t)\|$ has extreme values when $\frac{d}{dt}(\|\vec{r}(t)\|) = 0$

(at critical points)

but $\qquad \frac{d}{dt}\left(\|\vec{r}(t)\|\right) = \frac{d}{dt}\left(\sqrt{\vec{r}(t)\cdot\vec{r}(t)}\right)$

chain rule $\Rightarrow$ $\qquad = \frac{1}{2\sqrt{\vec{r}(t)\cdot\vec{r}(t)}}\left(\left(\vec{r}\,'(t)\cdot\vec{r}(t)\right) + \vec{r}(t)\cdot\vec{r}\,'(t)\right)$

$$= \frac{\vec{r}\,'(t)\cdot\vec{r}(t)}{\|\vec{r}(t)\|}$$

$$= \frac{\|\vec{r}\,'(t)\|\,\|\vec{r}(t)\|\cos\theta}{\|\vec{r}(t)\|}$$

$$= \|\vec{r}\,'(t)\|\cos\theta$$

when θ is the angle between $\vec{r}(t)$, $\vec{r}(t)$.

Thus, $\qquad \frac{d}{dt}\left(\|\vec{r}(t)\|\right) = 0 \Leftrightarrow \left(\cos\theta = 0 \text{ or } \|\vec{r}\,'(t)\| = 0\right) \Leftrightarrow \theta = 90^\circ.$

$\Rightarrow$ $\|\vec{r}(t)\|$ has extreme values at points where $\vec{r}(t) \perp \vec{r}\,'(t)$.