Notes from ~~[illegible]~~ §10.2

Important Identities!

$\cos(\cos^{-1}(t)) = t$ for any ~~real~~ t in $[-1,1]$

$\sin(\sin^{-1}(t)) = t$ for any ~~#s~~ t in $[-1,1]$

$\tan(\tan^{-1}(t)) = t$ for any real # t.

Ex: $\cos(\cos^{-1}(\frac{9}{10})) = \frac{9}{10}$

$\sin(\sin^{-1}(\frac{-7}{9})) = \frac{-7}{9}$

$\tan(\tan^{-1}(10000)) = 10000$.

Ex: $\cos(\cos^{-1}(2))$ is <u>undefined</u>

<u>why?</u> 2 is not in $[-1,1]$, so the identity does not work.

In fact $\cos^{-1}(2)$ is undefined.

(See definition of $\cos^{-1}(t)$).

These identities hold because $\cos^{-1}$, ~~&~~ $\sin^{-1}$, $\tan^{-1}$ are the inverse functions of cos, sin, tan ~~when~~ when you restrict the domains of cos, sin, tan approprietely (see definitions)

Warning: ~~the~~ If you change the order, you have to be careful

Identities:

If $0 \le \theta \le \pi$, then $\cos^{-1}(\cos(\theta)) = \theta$

If $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $\sin^{-1}(\sin(\theta)) = \theta$

If $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $\tan^{-1}(\tan(\theta)) = \theta$.

Notice: $\{\theta \mid 0 \le \theta \le \pi\}$ is the range of $\cos^{-1}$

$\{\theta \mid -\frac{\pi}{2} \le \theta \le \}$ is the range of $\sin^{-1}$

$\{\theta \mid -\frac{\pi}{2} < \theta < \frac{\pi}{2}\}$ is the range of $\tan^{-1}$

Ex: $\cos^{-1}(\cos(\frac{5\pi}{6})) = \frac{5\pi}{6}$

$\sin^{-1}(\sin(\frac{-\pi}{4})) = \frac{-\pi}{4}$

$\tan^{-1}(\tan(\frac{\pi}{3})) = \frac{\pi}{3}$

Be careful if θ is not in the correct interval!

Ex: $\sin^{-1}(\sin(\frac{2\pi}{3}))$ is <u>not</u> $\frac{2\pi}{3}$.

Why? $\sin(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2}$ (unit circle).

So $\sin^{-1}(\sin(\frac{2\pi}{3})) = \sin^{-1}(\frac{\sqrt{3}}{2})$.

Now $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is the angle θ such that

① $\sin(\theta) = \frac{\sqrt{3}}{2}$

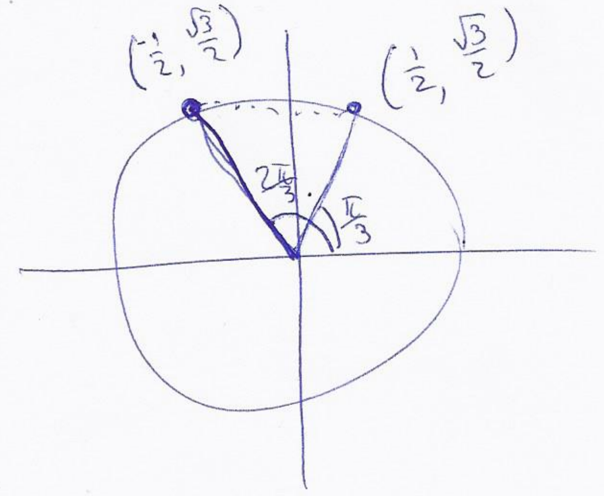
② $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$\therefore$ ~~Sin~~ $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

Summary: $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is the angle θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that

$$\sin(\theta) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

So $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$



← same y-value on the circle, but $\frac{\pi}{3}$ is in the right range.

Ex: $\cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right)$ is <u>not</u> $-\frac{\pi}{4}$.

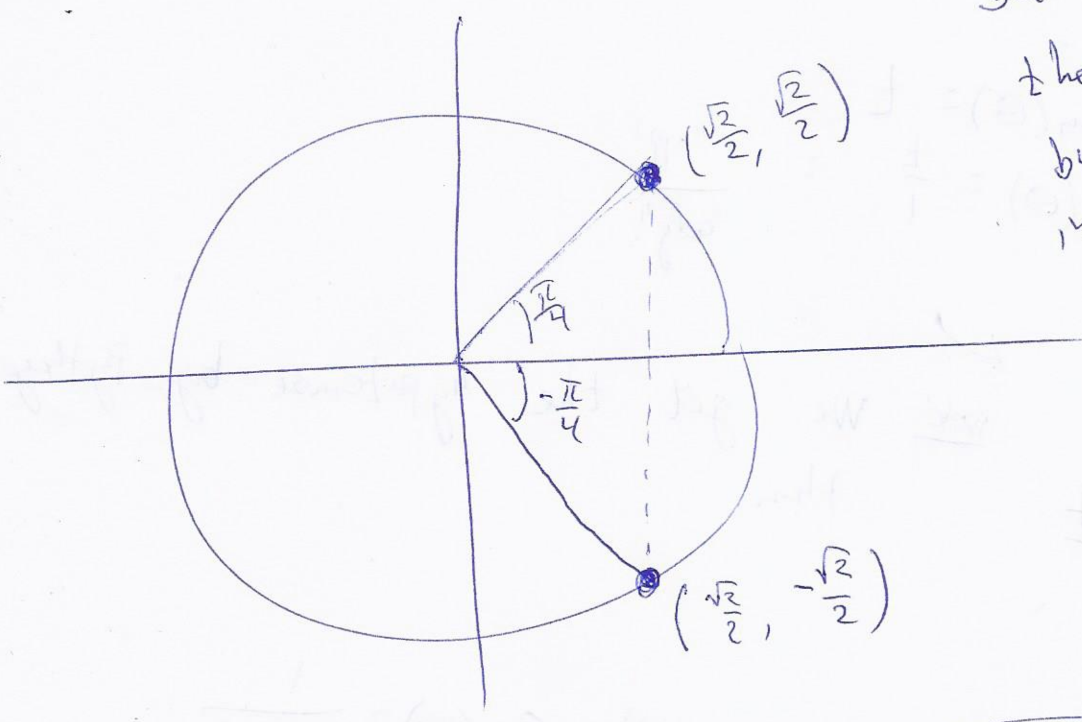
$\cos\left(-\frac{\pi}{4}\right) = +\frac{\sqrt{2}}{2}$ (unit circle)

So $\cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$.

Say $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \theta$. Then ① $\cos(\theta) = \frac{\sqrt{2}}{2}$ and

② $0 \le \theta \le \pi$.

So $\theta = \frac{\pi}{4}$



Same x-value on the unit circle but $\frac{\pi}{4}$ is in the right range.

Summary

$\cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right)$ is the angle θ in $[0, \pi]$ such that $\cos(\theta) = \cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Ex: Find $\sin(\tan^{-1}(2))$.

We will do this in two, essentially equivalent ways:

(1) Say $\tan^{-1}(2) = \theta$.

Then $\tan(\theta) = 2$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$\frac{\sin(\theta)}{\cos(\theta)} = 2$$

$$\boxed{\sin(\theta) = 2\cos(\theta)}$$

We also know:

$$\boxed{\sin^2(\theta) + \cos^2(\theta) = 1}$$

$$(2\cos(\theta))^2 + (\cos(\theta))^2 = 1$$

$$4\cos^2(\theta) + \cos^2(\theta) = 1$$

$$5\cos^2(\theta) = 1$$

$$\cos^2(\theta) = \frac{1}{5}$$

$$\cos(\theta) = \pm\frac{1}{\sqrt{5}}$$

Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\cos(\theta) > 0$

So $\cos(\theta) = \frac{1}{\sqrt{5}}$.

Since $\sin(\theta) = 2\cos(\theta)$

$$\sin(\theta) = 2\left(\frac{1}{\sqrt{5}}\right)$$

$$\sin(\theta) = \frac{2}{\sqrt{5}}$$

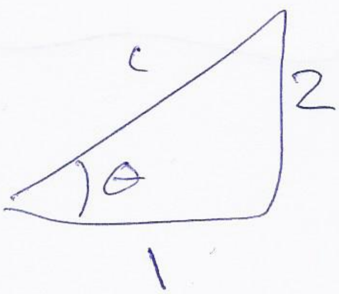
So $\sin(\tan^{-1}(2)) = \sin(\theta) = \frac{2}{\sqrt{5}}$

(2) To find $\sin(\tan^{-1}(2))$, we can also use triangles!

If $\tan^{-1}(2) = \theta$ then

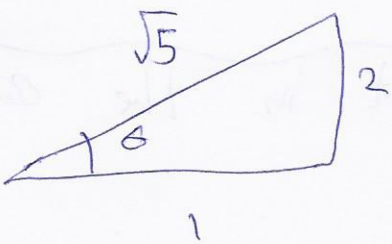
$\tan(\theta) = 2$

$\tan(\theta) = \frac{2}{1} = \frac{\text{"opp"}}{\text{"adj"}}$.



to get c, use

$c^2 = 1^2 + 4^2$



Then $\sin(\theta) = \frac{\text{"opp"}}{\text{"hyp"}}$

$\sin(\theta) = \frac{2}{\sqrt{5}}$

~~Sin~~

Think about why you know $\sin(\theta) > 0$.

Ex: Find a nice formula for $\sec(\tan^{-1}(t))$, where t is a real #.

Sol: Use triangles:

Suppose $\tan^{-1}(t) = \theta$. Then

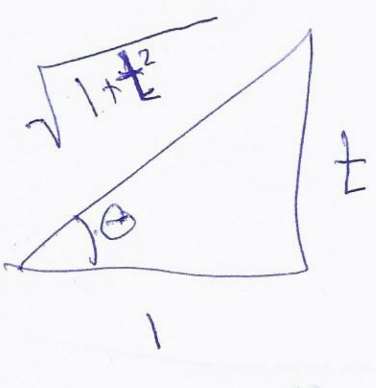
$\tan(\theta) = t$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Note: ① $\sec(\tan^{-1}(t)) = \sec(\theta) = \frac{1}{\cos(\theta)}$.

② Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\cos(\theta) > 0$.

$\tan(\theta) = t$

$\tan(\theta) = \frac{t}{1} = \frac{\text{"opp"}}{\text{"adj"}}$.



Note: We get the hypotenuse by Pythagorean thm.

Thus, $\sec(\tan^{-1}(t)) = \sec(\theta) = \frac{1}{\cos(\theta)}$

$= \frac{1}{\left(\frac{\text{"adj"}}{\text{"hyp"}}\right)}$

$= \frac{1}{\left(\frac{1}{\sqrt{1+t^2}}\right)}$

So $\sec(\tan^{-1}(t)) = \sqrt{1+t^2}$