Instructions:
1. Write your *Name, PID, Section Number,* and *Exam Version* on the *front* of your blue book.
2. You may use one $8.5x11$ in. sheet of *handwritten* notes, but no books or other assistance during this exam.
3. No calculator, phones, or any other electronic devices are allowed during this exam.
4. Present your solutions clearly in your Blue Book:
(a) Carefully indicate the number and letter of each question and each part of a question.
(b) Present your answers in the same order as they appear in the exam.
(c) Start each problem on a new page.
5. Show all of your work. Unsupported answers will receive no credit.
6. Turn in your exam paper and your note sheet with your Blue Book.

0. (1 point.) Carefully read and follow the instructions.

1. (4 points) Find all values of x such that

$$|5x-1| < 5.$$

Write your answer in interval notation.

Solution

If $5x-1 \geq 0$ (i.e., if the expression inside the absolute value is ≥ 0), then we have

$$\begin{aligned} |5x-1| &< 5 \\ 5x-1 &< 5 \\ 5x &< 6 \\ x &< \tfrac{6}{5} \end{aligned}.$$

If $5x-1 < 0$ (i.e., if the expression inside the absolute value is < 0), then we have

$$\begin{aligned} |5x-1| &< 5 \\ -(5x-1) &< 5 \\ 5x-1 &> -5 \\ 5x &> -4 \\ x &> \tfrac{-4}{5} \end{aligned}.$$

Therefore, if $|5x-1| < 5$, then $\frac{-4}{5} < x < \frac{6}{5}$. In interval notation, the set of all possible solutions is the interval $\left(\frac{-4}{5}, \frac{6}{5}\right)$.

2. Consider the parabola given by the equation $y = 5x^2 - 20x + 7$.

(a) (4 points) Rewrite this equation in the form $y = a(x+h)^2 + k$.

(b) (2 points) What is the vertex of this parabola?

(c) (2 points) For which values of x does this parabola intersect the x-axis?

Solution

(a) Complete the square:

$$\begin{aligned} y &= 5x^2 - 20x + 7 \\ &= 5(x^2 - 4x) + 7 \\ &= 5((x^2 - 4x + (-2)^2) - (-2)^2) + 7 \\ &= 5((x-2)^2 - 4) + 7 \\ &= 5(x-2)^2 - 20 + 7 \\ &= 5(x-2)^2 - 13 \end{aligned}.$$

(b) You could use the formula for the vertex: The x-coordinate of the vertex is $\frac{-b}{2a} = \frac{20}{(2)(5)} = 2$. Once you have the x-coordinate, plug it back into the equation to get $y = 5(2)^2 - 20(2) + 7 = 20 - 40 + 7 = -13$. So the vertex is $(2, -13)$
A much better option is to notice that the parabola is now in vertex form. So the vertex is $(-(-2), -13) = (2, -13)$.

(c) The parabola intersects the x-axis when $y = 0$. So we are trying to solve the equation $0 = 5x^2 - 20x + 7$. You could plug $a = 5$, $b = -20$, $c = 7$ into the quadratic equation so get

$$x = \frac{20 \pm \sqrt{(-20)^2 - (4)(5)(7)}}{(2)(5)} = \frac{10 \pm \sqrt{65}}{5}.$$

Since we already completed the square, it is faster to solve for x in the equation $0 = 5(x-2)^2 - 13$.

$$\begin{aligned} 0 &= 5(x-2)^2 - 13 \\ 13 &= 5(x-2)^2 \\ \tfrac{13}{5} &= (x-2)^2 \\ \pm\sqrt{\tfrac{13}{5}} &= x - 2 \\ 2 \pm \sqrt{\tfrac{13}{5}} &= x \end{aligned}.$$

3. Suppose the amount of wood in logs, L, that a woodchuck could chuck* is a linear function, $L = f(t)$, of the number of hours, t, she spends chucking wood. Suppose the woodchuck chucks 9 logs of wood in 2 hours, and 21 logs of wood in 5 hours.

   (a) (3 points) Find the formula for the linear function, $f(t) = mt + b$, that represents the amount of wood in logs that this woodchuck could chuck† after t hours.

   (b) (1 point) How much wood could this woodchuck chuck in 3 hours‡?

   (c) (3 points) Find the inverse, $f^{-1}(L)$, of the function f.

   (d) (1 point) How many hours would it take for this woodchuck to chuck 10 logs of wood?

*If, of course, a woodchuck *could* chuck wood.
†If this woodchuck could indeed chuck wood.
‡If this woodchuck could chuck wood.

Solution

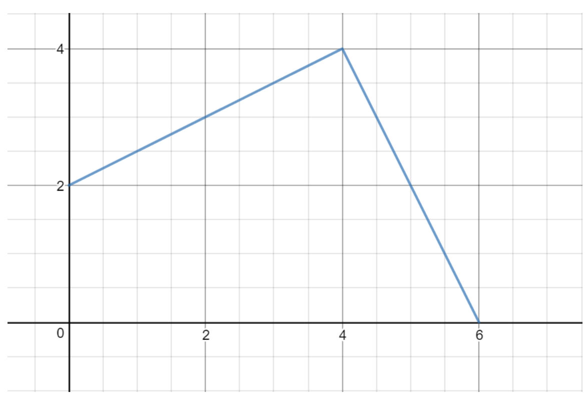
(a) we are looking for a linear function $f(t) = mt + b$ that represents this situation. Using the information from the problem, we see that $f(2) = 9$ and $f(5) = 21$. Then the slope is
$$m = \frac{21-9}{5-2} = \frac{12}{3} = 4.$$
To find b, plug in $t = 2$ or $t = 5$. For example: $f(2) = 9$, so $9 = 4(2) + b$ and therefore $b = 1$. Thus, $f(t) = 4t + 1$.

(b) The question is asking for $f(3) = 4(3) + 1 = 13$.

(c) To find the inverse, solve the equation $L = 4t + 1$. Since
$$t = \frac{L-1}{4},$$
we have
$$f^{-1}(L) = \frac{L-1}{4}.$$

(d) This question is asking for $f^{-1}(10) = \frac{10-1}{4} = \frac{9}{4}$.

4. Let $f(x)$ be the function whose graph is given below



Stetch the graphs of the following functions:

(a) (4 points) $g_1(x) = f(x-2) + 1$,

(b) (4 points) $g_2(x) = -\frac{1}{2}f(x) - 1$.

Solution

(a) Shift the graph up by 1 and to the right by 2.

(b) Reflect the graph about the x-axis, then stretch the graph vertically by a factor of 1/2 (i.e., shrink the graph vertically by a factor of 2), then shift down by 1 (order matters in part (b).)

5. Let
$$g(x) = 8(x+3)^3,$$
$$h(x) = \frac{\sqrt[3]{x}}{2} - 3.$$

(a) (3 points) Calculate $(h \circ g)(x)$

(b) (3 points) Calculate $(g \circ h)(x)$

(c) (2 points) What is the relationship between g and h?

Solution

(a)

$$\begin{aligned}(h \circ g)(x) &= h(8(x+3)^3) \\ &= \frac{\sqrt[3]{8(x+3)^3}}{2} - 3 \\ &= \frac{2(x+3)}{2} - 3 \\ &= (x+3) - 3 \\ &= x\end{aligned}.$$

(b)

$$\begin{aligned}(g \circ h)(x) &= g(\frac{\sqrt[3]{x}}{2} - 3) \\ &= 8((\frac{\sqrt[3]{x}}{2} - 3) + 3)^3 \\ &= 8(\frac{\sqrt[3]{x}}{2})^3 \\ &= 8(\frac{x}{8}) \\ &= x\end{aligned}.$$

(c) h and g are inverses of each other.

6. Let

$$f(x) = \frac{3x^2 - 3x - 6}{2x^2 - 6x + 4}.$$

(a) (2 points) Find all real numbers a such that $f(a) = 0$.

(b) (2 points) What is the domain of f?

(c) (2 points) Does the graph of $f(x)$ have any holes? If so, where?

(d) (2 points) Does the graph of $f(x)$ have any horizontal or vertical asymptotes? If so, where?

Solution

First factor the numerator and denominator

$$\frac{3x^2 - 3x - 6}{2x^2 - 6x + 4} = \frac{3(x-2)(x+1)}{2(x-2)(x-1)} = \frac{3(x+1)}{2(x-1)}.$$

(a) $f(a) = 0$ if and only if the numerator is zero. Therefore the only value that could possibly satisfies $f(a) = 0$ is $a = -1$. (Note: $a = 2$ is not in the domain of f. We did not penalize you if you also said $f(2) = 0$.)

(b) The domain of a rational function is all real numbers except where the denominator is zero. Since the denominator is zero at $x = 2$ and $x = 1$, the domain of f is the set of all real numbers x such that $x \neq 2, 1$.

(c) Since the term $x - 2$ factors out of both the numerator and the denominator, the graph has a hole at $x = 2$.

(d) Since $x - 1$ does not factor out of both the numerator and the denominator, the graph has a vertical asymptote at $x = 1$. Since the degree of the numerator is equal to the degree of the denominator, there is a horizontal asymptote at $y = \frac{3}{2}$ (the ratio of the leading coefficients.)