Solutions to Homework 2

Below are elements of solutions to the problems of Homework 2 marked for grading.

1.5 Problem 11  The components $x_2$, $x_4$ and $x_6$ can be taken as free variables. The system $Ax = 0$ leads to

$$
\begin{align*}
    x_5 &= 4x_6 \\
    x_3 &= x_6 \\
    x_1 &= 4x_2 + 2x_3 - 3x_5 + 5x_6 \\
         &= 4x_2 + 2x_6 - 3(4x_6) + 5x_6 \\
         &= 4x_2 - 5x_6.
\end{align*}
$$

Any solution is of the form

$$
    x = \begin{bmatrix}
    4x_2 - 5x_6 \\
    x_2 \\
    x_6 \\
    x_4 \\
    4x_6 \\
    x_6
\end{bmatrix}, \quad \text{where } x_2, x_4, \text{ and } x_6 \text{ vary freely in } \mathbb{R}.
$$

From this, it follows that the solution set of the homogeneous system $Ax = 0$ is

$$
    S = \left\{ \begin{bmatrix}
    4u - 5w \\
    u \\
    w \\
    v \\
    4w \\
    w
\end{bmatrix} : u, v, w, h \in \mathbb{R} \right\} = \left\{ u \begin{bmatrix}
    4 \\
    1 \\
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix} + v \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    1 \\
    0 \\
    1
\end{bmatrix} + w \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    1
\end{bmatrix} : u, v, w, h \in \mathbb{R} \right\}
$$

1.7 Problem 10  By row-reduction, the matrix $[v_1, v_2, v_3]$ is shown to be row-equivalent to the matrix:

$$
\begin{bmatrix}
    1 & -3 & 2 \\
    0 & 0 & 1 \\
    0 & 0 & h + 10
\end{bmatrix}.
$$

So:

(a) Because of the second row, the vector equation $xv_1 + yv_2 = v_3$ has no solution for any value of $h$. So $v_3$ never belongs to the linear span of $v_1$ and $v_2$.

(b) $\{v_1, v_2, v_3\}$ is always a linearly dependent set since $v_2 = -3v_1$.

1.8 Problem 30  Let $x$ be a vector in $\mathbb{R}^n$. Since $v_1, \ldots, v_p$ span $\mathbb{R}^n$, there are scalars $\alpha_1, \ldots, \alpha_p$ such that

$$
x = \alpha_1v_1 + \cdots + \alpha_pv_p.
$$

Then we have

$$
T(x) = T(\alpha_1v_1 + \cdots + \alpha_pv_p) \\
= \alpha_1T(x_1) + \ldots + \alpha_pT(x_p) \quad \text{(by linearity of } T) \\
= 0 \quad \text{(since } T(v_i) = 0 \text{ for all } i).
$$
1.9 Problem 18

\[ T(x_1, x_2) = (x_1 + 4x_2, 0, x_1 - 3x_2, x_1) = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \]

with \( A \) the \( 4 \times 2 \) matrix

\[
A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \\ 1 & -3 \\ 1 & 0 \end{bmatrix}.
\]