Problem 1: If $V$ is a $k$-vector space, recall that the dual space $V^*$ is the set of $k$-linear maps $\lambda : V \to k$ (such maps $\lambda$ are called linear functionals on $V$). The set $V^*$ is a $k$-vector space via the rule 
\[(\alpha_1 \lambda_1 + \alpha_2 \lambda_2)(v) := \alpha_1 \lambda_1(v) + \alpha_2 \lambda_2(v)\]
for all $\alpha_1, \alpha_2 \in k$, $\lambda_1, \lambda_2 \in V^*$, and $v \in V$. If $V$ is a $G$-module for some group $G$, prove that $V^*$ is a $G$-module by way of the rule 
\[(g \lambda)(v) := \lambda(g^{-1}v)\]
for all $g \in G$, $\lambda \in V^*$, and $v \in V$. Prove that if $V^*$ is irreducible, then $V$ is irreducible. Is $V$ necessarily isomorphic to $V^*$?

Problem 2: Let $G$ be a finite group and let $k[G]$ be the associated group algebra, over some field $k$. Let $Z$ be the center of $k[G]$; that is 
\[Z = \{z \in k[G] : za = az \text{ for all } a \in k[G]\} \tag{1}\]
Prove that $Z$ is a $k$-linear subspace of $k[G]$ and that 
\[\dim(Z) = \text{number of conjugacy classes of } G \tag{2}\]

Problem 3: Let $p$ be a prime and let $\mathbb{F}_p$ be the finite field with $p$ elements. Define a homomorphism $R : \mathbb{Z}/p\mathbb{Z} \to GL_2(\mathbb{F}_p)$ by 
\[\bar{a} \mapsto \begin{pmatrix} 1 & \bar{a} \\ 0 & 1 \end{pmatrix}, \tag{3}\]
for all $\bar{a} \in \mathbb{Z}/p\mathbb{Z}$. Let $V$ be the $\mathbb{F}_p$-vector space $(\mathbb{F}_p)^2$ with $\mathbb{Z}/p\mathbb{Z}$-action afforded by the map $R$. Prove that $V$ is indecomposable, but not irreducible, as a $\mathbb{Z}/p\mathbb{Z}$-module.

Problem 4: Let $G$ be a group acting on a finite set $S$. Let $k$ be a field and consider the $G$-module $k[S]$. The $G$-invariant subspace of $k[S]$ is 
\[k[S]^G := \{v \in k[S] : g.v = v \text{ for all } g \in G\} \tag{4}\]
Prove that the dimension of $k[S]^G$ is the number of orbits in the action of $G$ on $S$.

Problem 5: Let $V$ be a $k$-vector space (over some field $k$). A bilinear form is a function $B : V \times V \to k$ such that 
\[B(\alpha_1 v_1 + \alpha_2 v_2, \alpha_3 v_3 + \alpha_4 v_4) = \alpha_1 \alpha_3 B(v_1, v_3) + \alpha_1 \alpha_4 B(v_1, v_4) + \alpha_2 \alpha_3 B(v_2, v_3) + \alpha_2 \alpha_4 B(v_2, v_4)\]
for all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in k$ and $v_1, v_2, v_3, v_4 \in V$. The bilinear form $V$ is nondegenerate if, for any fixed nonzero $v \in V$, there exists $w \in V$ such that $B(v, w) \neq 0$. If $V$ is a $G$-module, a bilinear form $B$ on $V$ is $G$-invariant if $B(g.v, g.w) = B(v, w)$ for all $v, w \in V$ and $g \in G$. 


Let $V$ be a $G$-module over $k$ with a $G$-invariant inner product $V$. Endow the dual space $V^*$ with the structure of a $G$-module as in Problem 1. Prove that there is an injection of $G$-modules $\varphi : V \to V^*$.

**Problem 6:** Let $R : G \to GL_n(\mathbb{R})$ be a representation of a group $G$ over the real numbers. Since every real number is a complex number, we have an inclusion $GL_n(\mathbb{R}) \hookrightarrow GL_n(\mathbb{C})$. The composition $R_C : G \to GL_n(\mathbb{R}) \hookrightarrow GL_n(\mathbb{C})$ is the complexification of the representation $R$. Give an example of an irreducible representation $R$ whose complexification is not irreducible.

At the level of modules, consider a $G$-module $V$ over $\mathbb{R}$. The complexification $V_C$ of $V$ is the tensor product $V_C := \mathbb{C} \otimes_\mathbb{R} V$. This is less scary than it seems; if $V$ has basis $B$, we may identify $V \cong \mathbb{R}[B]$ and also identify $V_C \cong \mathbb{C}[B]$ – the complexification $V_C$ ‘extends scalars’ from $\mathbb{R}$ to $\mathbb{C}$. The complexification $V_C$ becomes a $G$-module by the rule

$$g.(\alpha_1 b_1 + \cdots + \alpha_n b_n) := \alpha_1 (g . b_1) + \cdots + \alpha_n (g . b_n)$$

for all $g \in G, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$, and $b_1, \ldots, b_n \in B$.

**Problem 7:** Let $G$ be a group acting on a finite set $S$. Let $k[S]$ be the associated $G$-module (over a field $k$) and let $\chi : G \to k$ be the associated character. Prove that, for $g \in G$,

$$\chi(g) = \text{number of fixed points in the action of } g \text{ on } S.$$

**Problem 8:** (Optional - not to be handed in.) Suppose we have a $6 \times 6$ complex matrix $A$ with two eigenvalues, 3 and 4. We know that $\text{rank}(A - 3I) = 4$ and $\text{rank}(A - 4I) = 5$. Write down all the possibilities for the Jordan Canonical Form of the matrix $A$. If in addition we know that $\text{rank}(A - 3I)^2 = 2$, determine the Jordan Canonical Form of $A$.

**Problem 9:** (Optional - not to be handed in.) Let $G$ be a group acting on a set $S$. The kernel $K$ of the action of $G$ on $S$ is the subset of $G$:

$$K := \{ g \in G : g.s = s \text{ for all } s \in S \}.$$

Prove that $K$ is a normal subgroup of $G$ and explain why $G/K$ naturally acts on $S$, with the same orbits as the action of $G$ on $S$. (The operation $G \sim G/K$ eliminates ‘fuzz’ in our action.)

**Problem 10:** (Optional - not to be handed in.) Let $G$ be a group. For $x, y \in G$, the commutator is $[x, y] := xyx^{-1}y^{-1} \in G$. The derived subgroup $G'$ of $G$ is the subgroup generated by all possible commutators of elements in $G$:

$$G' = \langle [x, y] : x, y \in G \rangle.$$

Prove the following.

1. $G' = 1$ if and only if $G$ is abelian.
2. $G'$ is a normal subgroup of $G$.
3. The quotient $G/G'$ is abelian (this quotient is called the abelianization $G^{ab}$ of $G$).
(4) If $N \subseteq G$ is any normal subgroup with $G/N$ abelian, then $G' \subseteq N$.

(5) If $A$ is any abelian group and $\varphi : G \to A$ is a group homomorphism then $G' \subseteq \ker(\varphi)$.

Calculate the derived subgroup of the dihedral group $D_n$ and the quaternion group $Q_8$.

A group $G$ is perfect if $G'' = G$; find an example of a perfect group.