Math 100C: Spring 2017
Homework 2
Due 4/19/2017

Problem 1: If V is a k-vector space, recall that the *dual space* V^* is the set of k-linear maps $\lambda : V \to k$ (such maps λ are called *linear functionals* on V). The set V^* is a k-vector space via the rule

$$(\alpha_1\lambda_1 + \alpha_2\lambda_2)(v) := \alpha_1\lambda_1(v) + \alpha_2\lambda_2(v)$$

for all $\alpha_1, \alpha_2 \in k$, $\lambda_1, \lambda_2 \in V^*$, and $v \in V$. If V is a G-module for some group G, prove that V^* is a G-module by way of the rule

$$(g.\lambda)(v) := \lambda(g^{-1}.v)$$

for all $g \in G, \lambda \in V^*$, and $v \in V$. Prove that if V^* is irreducible, then V is irreducible. Is V necessarily isomorphic to V^*?

Problem 2: Let G be a finite group and let $k[G]$ be the associated group algebra, over some field k. Let Z be the *center* of $k[G]$; that is

$$Z = \{z \in k[G] \,:\, za = az \text{ for all } a \in k[G]\ \}.$$

Prove that Z is a k-linear subspace of $k[G]$ and that

$$\dim(Z) = \text{number of conjugacy classes of } G.$$

Problem 3: Let p be a prime and let $\mathbb{F}_p$ be the finite field with p elements. Define a homomorphism $R : \mathbb{Z}/p\mathbb{Z} \to GL_2(\mathbb{F}_p)$ by

$$\overline{n} \mapsto \begin{pmatrix} \overline{1} & \overline{n} \\ \overline{0} & \overline{1} \end{pmatrix},$$

for all $\overline{n} \in \mathbb{Z}/p\mathbb{Z}$. Let V be the $\mathbb{F}_p$-vector space $(\mathbb{F}_p)^2$ with $\mathbb{Z}/p\mathbb{Z}$-action afforded by the map R. Prove that V is indecomposable, but not irreducible, as a $\mathbb{Z}/p\mathbb{Z}$-module.

Problem 4: Let G be a group acting on a finite set S. Let k be a field and consider the G-module $k[S]$. The *G-invariant subspace* of $k[S]$ is

$$k[S]^G := \{v \in k[S] \,:\, g.v = v \text{ for all } g \in G\ \}.$$

Prove that the dimension of $k[S]^G$ is the number of orbits in the action of G on S.

Problem 5: Let V be a k-vector space (over some field k). A *bilinear form* is a function $B : V \times V \to k$ such that

$$B(\alpha_1 v_1 + \alpha_2 v_2, \alpha_3 v_3 + \alpha_4 v_4) = \alpha_1\alpha_3 B(v_1, v_3) + \alpha_1\alpha_4 B(v_1, v_4) + \alpha_2\alpha_3 B(v_2, v_3) + \alpha_2\alpha_4 B(v_2, v_4)$$

for all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in k$ and $v_1, v_2, v_3, v_4 \in V$. The bilinear form V is *nondegenerate* if, for any fixed nonzero $v \in V$, there exists $w \in V$ such that $B(v, w) \neq 0$. If V is a G-module, a bilinear form B on V is *G-invariant* if $B(g.v, g.w) = B(v, w)$ for all $v, w \in V$ and $g \in G$.

Let V be a G-module over k with a G-invariant inner product V. Endow the dual space V^* with the structure of a G-module as in Problem 1. Prove that there is an injection of G-modules $\varphi : V \to V^*$.

Problem 6: Let $R : G \to GL_n(\mathbb{R})$ be a representation of a group G over the real numbers. Since every real number is a complex number, we have an inclusion $GL_n(\mathbb{R}) \hookrightarrow GL_n(\mathbb{C})$. The composition

$$R^{\mathbb{C}} : G \to GL_n(\mathbb{R}) \hookrightarrow GL_n(\mathbb{C})$$

is the *complexification* of the representation R. Give an example of an irreducible representation R whose complexification is not irreducible.

At the level of modules, consider a G-module V over $\mathbb{R}$. The *complexification* $V_{\mathbb{C}}$ of V is the tensor product $V_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} V$. This is less scary than it seems; if V has basis $\mathcal{B}$, we may identify $V \cong \mathbb{R}[\mathcal{B}]$ and also identify $V_{\mathbb{C}} \cong \mathbb{C}[\mathcal{B}]$ – the complexification $V_{\mathbb{C}}$ 'extends scalars' from $\mathbb{R}$ to $\mathbb{C}$. The complexification $V_{\mathbb{C}}$ becomes a G-module by the rule

$$g.(\alpha_1 b_1 + \cdots + \alpha_n b_n) := \alpha_1(g.b_1) + \cdots + \alpha_n(g.b_n)$$

for all $g \in G, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$, and $b_1, \ldots, b_n \in \mathcal{B}$.

Problem 7: Let G be a group acting on a finite set S. Let $k[S]$ be the associated G-module (over a field k) and let $\chi : G \to k$ be the associated character. Prove that, for $g \in G$,

$$\chi(g) = \text{number of fixed points in the action of } g \text{ on } S.$$

Problem 8: (Optional - not to be handed in.) Suppose we have a 6×6 complex matrix A with two eigenvalues, 3 and 4. We know that $\operatorname{rank}(A-3I) = 4$ and $\operatorname{rank}(A-4I) = 5$. Write down all the possibilities for the Jordan Canonical Form of the matrix A. If in addition we know that $\operatorname{rank}(A-3I)^2 = 2$, determine the Jordan Canonical Form of A.

Problem 9: (Optional - not to be handed in.) Let G be a group acting on a set S. The *kernel* K of the action of G on S is the subset of G:

$$K := \{g \in G \,:\, g.s = s \text{ for all } s \in S\}.$$

Prove that K is a normal subgroup of G and explain why G/K naturally acts on S, with the same orbits as the action of G on S. (The operation $G \rightsquigarrow G/K$ eliminates 'fuzz' in our action.)

Problem 10: (Optional - not to be handed in.) Let G be a group. For $x, y \in G$, the *commutator* is $[x, y] := xyx^{-1}y^{-1} \in G$. The *derived subgroup* G' of G is the subgroup generated by all possible commutators of elements in G:

$$G' = \langle [x, y] \,:\, x, y \in G \rangle.$$

Prove the following.

(1) $G' = 1$ if and only if G is abelian.
(2) G' is a normal subgroup of G.
(3) The quotient G/G' is abelian (this quotient is called the *abelianization* G^{ab} of G).

(4) If $N \subseteq G$ is any normal subgroup with G/N abelian, then $G' \subseteq N$.
(5) If A is any abelian group and $\varphi : G \to A$ is a group homomorphism then $G' \subseteq \ker(A)$.

Calculate the derived subgroup of the dihedral group D_n and the quaternion group Q_8. A group G is *perfect* if $G' = G$; find an example of a perfect group.