Problem 1: A fixed point of a permutation $\pi \in S_n$ is a number $1 \leq i \leq n$ such that $\pi(i) = i$. Let $\text{fix}(\pi)$ be the number of fixed points of a permutation $\pi$. If $n > 1$, use representation theory to prove that 
\[ \sum_{\pi \in S_n} (\text{fix}(\pi))^2 = 2(n!). \]

Problem 2: Let $G$ and $H$ be finite groups and let $G \times H$ be the product group. Prove that the number of conjugacy classes of $G \times H$ is the number of conjugacy classes of $G$ times the number of conjugacy classes of $H$. If $R_1, \ldots, R_s$ is a complete list of the non-isomorphic irreducible matrix representations of $G$ over $C$ and $S_1, \ldots, S_t$ is a complete list of the non-isomorphic irreducible matrix representations of $H$ over $C$, prove that 
\[ \{ R_i \otimes S_j : 1 \leq i \leq s, 1 \leq j \leq t \} \]
is a complete list of the non-isomorphic matrix representations for $G \times H$ over $C$ (see Homework 1 for the definition of the tensor matrix representation $R_i \otimes S_j$).

Problem 3: Let $G$ be a finite abelian group. Classify the irreducible matrix representations of $G$ over $C$. (Hint: Use the structure theorem for finite abelian groups, the classification in Homework 2 for cyclic groups, and Problem 2.)

Problem 4: Let $G$ be a finite group. For $x, y \in G$ the commutator of $x$ and $y$ is $[x, y] := xyx^{-1}y^{-1} \in G$. The derived subgroup is the subgroup of $G$ generated by all possible commutators:
\[ G' := \langle [x, y] : x, y \in G \rangle \]
Prove that $G'$ is a normal subgroup of $G$ and that the number of distinct 1-dimensional complex representations of $G$ is the cardinality $|G/G'|$.  

Problem 5: Let $Q_8 := \{ \pm 1, \pm i, \pm j, \pm k \}$ be the quaternion group. Determine the character table of $Q_8$.

Problem 6: Let $D_4 := \langle r, s \mid r^4 = s^2 = 1, srs = r^{-1} \rangle$ denote the group of symmetries of the square. Find the character table of $D_4$. Let $S$ be the 6-element set of diagonals and sides in the square (2 diagonals, 4 sides). Let $V$ be the corresponding 6-dimensional permutation representation of $D_4$. Calculate the decomposition of $V$ into irreducibles.

Problem 7: Let $G$ be a finite group and let $V$ be an irreducible (finite-dimensional) representation of $G$ over an algebraically closed field $k$. Suppose that $B_1 : V \times V \to k$ and $B_2 : V \times V \to k$ are two $G$-invariant bilinear forms on $V$. Prove that there exists a field element $\gamma \in k$ such that 
\[ B_1(v, w) = \gamma \cdot B_2(v, w) \]

\[ \text{The quotient } G/G' \text{ is called the abelianization } G^{ab} \text{ of } G; \text{ it is the maximal abelian quotient of } G. \]
for all \(v, w \in V\). (Hint: Consider the dual representation \(V^*\), which is also irreducible by Homework 2. We have maps \(\varphi_i : V \to V^*\) of \(G\)-modules given by \(\varphi_i(v)(w) := B_i(v, w)\) for \(i = 1, 2\). Apply Schur’s Lemma.)

**Problem 8:** (Optional - not to be handed in.) Suppose we have a \(6 \times 6\) complex matrix \(A\) with two eigenvalues, 3 and 4. We know that \(\text{rank}(A - 3I) = 4\) and \(\text{rank}(A - 4I) = 5\). Write down all the possibilities for the Jordan Canonical Form of the matrix \(A\). If in addition we know that \(\text{rank}(A - 3I)^2 = 2\), determine the Jordan Canonical Form of \(A\).

**Problem 9:** (Optional - not to be handed in.) Let \(G\) be a finite group and let \(S\) and \(T\) be finite \(G\)-sets. Consider the two following properties, which may or may not hold.

1. There is a bijection \(\psi : S \to T\) such that \(\psi(g.s) = g.\psi(s)\) for all \(g \in G\) and \(s \in S\).
2. There is a \(\mathbb{C}\)-linear bijective map \(\varphi : \mathbb{C}[S] \to \mathbb{C}[T]\) with the property that \(\varphi(g.v) = g.\varphi(v)\) for all \(g \in G\) and \(v \in \mathbb{C}[S]\).

Thus Property 2 just says that the permutation \(G\)-modules \(\mathbb{C}[S]\) and \(\mathbb{C}[T]\) are isomorphic. Prove that Property 1 implies Property 2, but Property 2 does not necessarily imply Property 1.

**Problem 10:** (Optional - not to be handed in.) What about representations of infinite groups? Let’s look at the example of the circle group

\[ S^1 = \{\exp(\theta i) : \theta \in \mathbb{R}\} = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}. \]

As is typical in this setting, we will only consider complex representations \(R : S^1 \to GL_n(\mathbb{C})\) which are continuous (in the Euclidean topologies on \(S^1\) and \(GL_n(\mathbb{C})\) – if you don’t know what this means, just interpret ‘continuous’ as in calculus).

- Let \(R : S^1 \to GL_1(\mathbb{C}) \cong \mathbb{C}^\times\) be a nontrivial 1-dimensional representation of \(S^1\). Prove that there is a unique minimum value of \(\theta_0\) in the real interval \((0, 2\pi]\) such that \(R(\exp(\theta_0 i)) = 1\). (Hint: Since \(R\) is a group homomorphism, if no such \(\theta_0\) existed, prove that there is a dense subset \(D\) of \(S^1\) such that \(R(z) = 1\) for all \(z \in D\). Since \(R\) is continuous, this implies that \(R\) is trivial.)

- Convince yourself that a complete set of the 1-dimensional representations of \(S^1\) is given by

\[ \{R_n : S^1 \to GL_1(\mathbb{C}) : n \in \mathbb{Z}\}, \]

where \(R_n : z \mapsto z^n\). (Hint: Use the previous part and consider the possible values for \(\theta_0\). Given that \(R\) is continuous and a homomorphism, what could \(R\) possibly do to the circular arc \(\{\exp(\theta i) : 0 \leq \theta \leq \theta_0\}\)? Don’t worry too much if you haven’t seen enough math to give a rigorous argument here – your intuition is correct!)

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\(^2\)This exercise can be modified to show that, if \(V\) is an irreducible representation of \(G\) over \(\mathbb{C}\), then a \(G\)-invariant inner product on \(V\) is unique up to a scalar. The argument has to be modified slightly due to the sequilinearity of an inner product in its second argument.

\(^3\)This was the subject of an awkward exchange between me and a referee for a journal article of mine, intermediated by the journal editor.