Math 100C: Spring 2017
Homework 3
Due 4/26/2017

Problem 1: A *fixed point* of a permutation $\pi \in S_n$ is a number $1 \leq i \leq n$ such that $\pi(i) = i$. Let $\mathrm{fix}(\pi)$ be the number of fixed points of a permutation π. If $n > 1$, use representation theory to prove that

$$\sum_{\pi \in S_n} (\mathrm{fix}(\pi))^2 = 2(n!).$$

Problem 2: Let G and H be finite groups and let $G \times H$ be the product group. Prove that the number of conjugacy classes of $G \times H$ is the number of conjugacy classes of G times the number of conjugacy classes of H. If $R_1, \ldots, R_s$ is a complete list of the non-isomorphic irreducible matrix representations of G over $\mathbb{C}$ and $S_1, \ldots, S_t$ is a complete list of the non-isomorphic irreducible matrix representations of H over $\mathbb{C}$, prove that

$$\{R_i \otimes S_j \,:\, 1 \leq i \leq s, 1 \leq j \leq t\}$$

is a complete list of the non-isomorphic matrix representations for $G \times H$ over $\mathbb{C}$ (see Homework 1 for the definition of the tensor matrix representation $R_i \otimes S_j$).

Problem 3: Let G be a finite abelian group. Classify the irreducible matrix representations of G over $\mathbb{C}$. (Hint: Use the structure theorem for finite abelian groups, the classification in Homework 2 for cyclic groups, and Problem 2.)

Problem 4: Let G be a finite group. For $x, y \in G$ the *commutator* of x and y is $[x, y] := xyx^{-1}y^{-1} \in G$. The *derived subgroup* is the subgroup of G generated by all possible commutators:

$$G' := \langle [x, y] \,:\, x, y \in G \rangle$$

Prove that G' is a normal subgroup of G and that the number of distinct 1-dimensional complex representations of G is the cardinality $|G/G'|$. [1]

Problem 5: Let $Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group. Determine the character table of Q_8.

Problem 6: Let $D_4 := \langle r, s \mid r^4 = s^2 = 1, srs = r^{-1} \rangle$ denote the group of symmetries of the square. Find the character table of D_4. Let S be the 6-element set of diagonals and sides in the square (2 diagonals, 4 sides). Let V be the corresponding 6-dimensional permutation representation of D_4. Calculate the decomposition of V into irreducibles.

Problem 7: Let G be a finite group and let V be an irreducible (finite-dimensional) representation of G over an algebraically closed field k. Suppose that $B_1 : V \times V \to k$ and $B_2 : V \times V \to k$ are two G-invariant bilinear forms on V. Prove that there exists a field element $\gamma \in k$ such that

$$B_1(v, w) = \gamma \cdot B_2(v, w)$$

[1] The quotient G/G' is called the *abelianization* G^{ab} of G; it is the maximal abelian quotient of G.

for all $v, w \in V$. (Hint: Consider the dual representation V^*, which is also irreducible by Homework 2. We have maps $\varphi_i : V \to V^*$ of G-modules given by $\varphi_i(v)(w) := B_i(v, w)$ for $i = 1, 2$. Apply Schur's Lemma.) [2]

Problem 8: (Optional - not to be handed in.) Suppose we have a 6×6 complex matrix A with two eigenvalues, 3 and 4. We know that $\text{rank}(A-3I) = 4$ and $\text{rank}(A-4I) = 5$. Write down all the possibilities for the Jordan Canonical Form of the matrix A. If in addition we know that $\text{rank}(A-3I)^2 = 2$, determine the Jordan Canonical Form of A.

Problem 9: (Optional - not to be handed in.) Let G be a finite group and let S and T be finite G-sets. Consider the two following properties, which may or may not hold.

(1) There is a bijection $\psi : S \to T$ such that $\psi(g.s) = g.\psi(s)$ for all $g \in G$ and $s \in S$.
(2) There is a $\mathbb{C}$-linear bijective map $\varphi : \mathbb{C}[S] \to \mathbb{C}[T]$ with the property that $\varphi(g.v) = g.\varphi(v)$ for all $g \in G$ and $v \in \mathbb{C}[S]$.

Thus Property 2 just says that the permutation G-modules $\mathbb{C}[S]$ and $\mathbb{C}[T]$ are isomorphic. Prove that Property 1 implies Property 2, but Property 2 does not necessarily imply Property 1. [3]

Problem 10: (Optional - not to be handed in.) What about representations of *infinite* groups? Let's look at the example of the *circle group*

$$S^1 = \{\exp(\theta i) \,:\, \theta \in \mathbb{R}\} = \{z \in \mathbb{C} \,:\, |z| = 1\} \subset \mathbb{C}.$$

As is typical in this setting, we will only consider complex representations $R : S^1 \to GL_n(\mathbb{C})$ which are *continuous* (in the Euclidean topologies on S^1 and $GL_n(\mathbb{C})$ – if you don't know what this means, just interpret 'continuous' as in calculus).

- Let $R : S^1 \to GL_1(\mathbb{C}) \cong \mathbb{C}^\times$ be a nontrivial 1-dimensional representation of S^1. Prove that there is a unique minimum value of θ_0 in the real interval $(0, 2\pi]$ such that $R(\exp(\theta_0 i) = 1$. (Hint: Since R is a group homomorphism, if no such θ_0 existed, prove that there is a dense subset D of S^1 such that $R(z) = 1$ for all $z \in D$. Since R is continuous, this implies that R is trivial.)
- Convince yourself that a complete set of the 1-dimensional representations of S^1 is given by
$$\{R_n : S^1 \to GL_1(\mathbb{C}) \,:\, n \in \mathbb{Z}\},$$
where $R_n : z \mapsto z^n$. (Hint: Use the previous part and consider the possible values for θ_0. Given that R is continuous and a homomorphism, what could R possibly do to the circular arc $\{\exp(\theta i) \,:\, 0 \le \theta \le \theta_0\}$? Don't worry too much if you haven't seen enough math to give a rigorous argument here – your intuition is correct!)

[2] This exercise can be modified to show that, if V is an irreducible representation of G over $\mathbb{C}$, then a G-invariant inner product on V is unique up to a scalar. The argument has to be modified slightly due to the sequilinearity of an inner product in its second argument.

[3] This was the subject of an awkward exchange between me and a referee for a journal article of mine, intermediated by the journal editor.