Problem 1: Let $F$ be a field of characteristic zero and let $f \in F[x]$ be a monic polynomial with derivative $f'$. Let $g \in F[x]$ be an irreducible polynomial with $g \mid f$ and $g \mid f'$. Prove that $g^2 \mid f$.

Problem 2: Let $p$ be a prime, let $r$ be a positive integer, and let $q = p^r$. To what finite abelian group is the additive group $F_q$ of the $q$-element field $F_q$ isomorphic?

Problem 3: Factor the polynomials $x^9 - x$ and $x^{27} - x$ over the field $F_3$.

Problem 4: Let $p$ be a prime number, let $r \geq 1$, and let $q = p^r$. Let $K$ be a field of order $q$. Define $\sigma : K \to K$ by $\sigma(x) = x^p$. Prove the following:

1. $\sigma : K \to K$ is a field automorphism.
2. $\sigma$ has order $r$.
3. $\sigma$ generates the group $\text{Aut}(K)$ of field automorphisms of $K$.

Problem 5: Prove that every finite extension of a finite field has a primitive element.

Problem 6: Let $F$ be a finite field and let $f(x) \in F[x]$ be a nonconstant polynomial with $f'(x) = 0$. Prove that $f(x)$ is not irreducible in $F[x]$.

Problem 7: The following polynomials are symmetric in the variables $x_1, x_2, x_3$. Write them in terms of the elementary symmetric polynomials $s_1, s_2, s_3$.

\[
\begin{align*}
f(x) &= x_1^3 + x_2^3 + x_3^3 \\
g(x) &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2
\end{align*}
\]

Problem 8: (Optional - not to be handed in.) Is there an analog of character theory for $G$-sets? Let $G$ be a finite group and let $S$ and $T$ be finite $G$-sets. We say that $S$ and $T$ are isomorphic as $G$-sets if there is a bijection $\phi : S \to T$ such that $\phi(g.s) = g.\phi(s)$ for all $g \in G$ and $s \in S$.

1. Given $s \in S$, let $G_s := \{g \in G : g.s = s\}$ be the stabilizer of $S$. Prove that for any $g \in G$ and $s \in S$ we have $G_{g.s} = gG_s g^{-1}$.
2. Prove that every orbit of the action of $G$ on $S$ is isomorphic to the action of $G$ on the set of left cosets $G/H$ given by $g(g'H) := (gg')H$, for some subgroup $H \subseteq G$.
3. Prove that $S$ and $T$ are isomorphic as $G$-sets if and only if the following condition holds:

   for any subgroup $H \subseteq G$ which is the stabilizer of some element of $S$ or of $T$, we have $|S^H| = |T^H|$, where $S^H = \{s \in S : h.s = s$ for all $h \in H\}$ and similarly for $T^H$.

This criterion for determining $G$-set isomorphism is equivalent to the Frobenius table of marks. Explain why it is impractical for $G = S_n$ in general. The table of marks is not often used.
Problem 9: (Optional - not to be handed in.) **Restriction of representations.**
Let $G$ be a finite group and let $H$ be a subgroup of $G$. If $R : G \to GL_n(k)$ is a matrix representation of $G$, the restriction of $R$ to $H$ is the function

$$R \downarrow_H : H \to GL_n(k)$$

obtained by restricting $R$ to $H \subseteq G$.

1. If $V$ is a $G$-module, what is the restriction $V \downarrow_H$ (i.e., how can we think of it)?
2. It $\chi : G \to k$ is the character of a matrix representation $R$ of $G$, what is the character $\chi \downarrow_H : H \to k$ of $R \downarrow_H$?
3. Show that if $V$ is an irreducible $G$-module, then $V \downarrow_H$ is not necessarily an irreducible $H$-module.

Problem 10: (Optional - not to be handed in.) **Induction of representations.** Let $G$ be a finite group and let $G \subseteq H$ be a subgroup of index $[G : H] = r$. Let $R : H \to GL_n(k)$ be a representation of $H$. We define a representation $R \uparrow^G : G \to GL_{nr}(k)$ (the induction of $R$ to $G$) as follows. Let $T = \{t_1, t_2, \ldots, t_r\}$ be a left transversal for $H$ in $G$, so that every element $g \in G$ may be written uniquely as $g = t_i h$ for some $1 \leq i \leq r$ and $h \in H$. Given $g \in G$, we define $R \uparrow^G (g)$ to be the block matrix

$$R \uparrow^G (g) := \begin{pmatrix}
R(t_1^{-1}gt_1) & \cdots & R(t_r^{-1}gt_1) \\
\vdots & & \vdots \\
R(t_1^{-1}gt_r) & \cdots & R(t_r^{-1}gt_r)
\end{pmatrix},$$

where we interpret $R(t_i^{-1}gt_j) = 0$ if $t_i^{-1}gt_j \notin H$.

1. Prove that for any $g \in G$, the matrix $R \uparrow^G (g)$ is a ‘block permutation matrix’ – just one of the blocks in every row and column above is nonzero. Also show that $R \uparrow^G (1)$ is the identity matrix in $GL_{nr}(k)$.
2. Prove that for any $x, y \in G$ we have $R \uparrow^G (xy) = R \uparrow^G (x)R \uparrow^G (y)$, so that $R \uparrow^G$ is actually a representation of $G$.
3. Let $\chi : H \to k$ be the character of $R$. Prove that the character $\chi \uparrow^G$ of $R \uparrow^G$ is given by the formula

$$\chi \uparrow^G (g) = \sum_{i=1}^r \chi(t_i^{-1}gt_i),$$

where we make the convention that $\chi(x) = 0$ if $x \notin H$.
4. Now assume $k = \mathbb{C}$. Prove that $\chi \uparrow^G (x)$ is also given by the formula

$$\chi \uparrow^G (g) = \frac{1}{|H|} \sum_{x \in G} \chi(xgx^{-1}).$$

Deduce that the representation $R \uparrow^G$ does not depend on the choice of the transversal $T$, up to isomorphism.
5. Give an example of an irreducible matrix representation $R$ of $H$ over $\mathbb{C}$ such that $R \uparrow^G$ is not irreducible.