Math 100C: Spring 2017
Homework 5
Due 5/17/2017

Problem 1: Let F be a field of characteristic zero and let $f \in F[x]$ be a monic polynomial with derivative f'. Let $g \in F[x]$ be an *irreducible* polynomial with $g \mid f$ and $g \mid f'$. Prove that $g^2 \mid f$.

Problem 2: Let p be a prime, let r be a positive integer, and let $q = p^r$. To what finite abelian group is the additive group $\mathbb{F}_q^+$ of the q-element field $\mathbb{F}_q$ isomorphic?

Problem 3: Factor the polynomials $x^9 - x$ and $x^{27} - x$ over the field $\mathbb{F}_3$.

Problem 4: Let p be a prime number, let $r \geq 1$, and let $q = p^r$. Let K be a field of order q. Define $\sigma : K \to K$ by $\sigma(x) = x^p$. Prove the following:

(1) $\sigma : K \to K$ is a field automorphism.
(2) σ has order r.
(3) σ generates the group $\mathrm{Aut}(K)$ of field automorphisms of K.

Problem 5: Prove that every finite extension of a finite field has a primitive element.

Problem 6: Let F be a finite field and let $f(x) \in F[x]$ be a nonconstant polynomial with $f'(x) = 0$. Prove that $f(x)$ is not irreducible in $F[x]$.

Problem 7: The following polynomials are symmetric in the variables x_1, x_2, x_3. Write them in terms of the elementary symmetric polynomials s_1, s_2, s_3.

$$f(x) = x_1^3 + x_2^3 + x_3^3$$
$$g(x) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

Problem 8: (Optional - not to be handed in.) Is there an analog of character theory for G-sets? Let G be a finite group and let S and T be finite G-sets. We say that S and T are *isomorphic* as G-sets if there is a bijection $\phi : S \to T$ such that $\phi(g.s) = g.\phi(s)$ for all $g \in G$ and $s \in S$.

(1) Given $s \in S$, let $G_s := \{g \in G \,:\, g.s = s\}$ be the stabilizer of S. Prove that for any $g \in G$ and $s \in S$ we have $G_{g.s} = gG_sg^{-1}$.
(2) Prove that every orbit of the action of G on S is isomorphic to the action of G on the set of left cosets G/H given by $g(g'H) := (gg')H$, for some subgroup $H \subseteq G$.
(3) Prove that S and T are isomorphic as G-sets if and only if the following condition holds:

    for any subgroup $H \subseteq G$ which is the stabilizer of some element of S or of T, we have $|S^H| = |T^H|$, where $S^H = \{s \in S \,:\, h.s = s \text{ for all } h \in H\}$ and similarly for T^H.

This criterion for determining G-set isomorphism is equivalent to the *Frobenius table of marks*. Explain why it is impractical for $G = S_n$ in general. The table of marks is not often used.

Problem 9: (Optional - not to be handed in.) **Restriction of representations.** Let G be a finite group and let H be a subgroup of G. If $R : G \to GL_n(k)$ is a matrix representation of G, the *restriction of R to H* is the function

$$R \downarrow_H : H \to GL_n(k)$$

obtained by restricting R to $H \subseteq G$.

(1) If V is a G-module, what is the restriction $V \downarrow_H$ (i.e., how can we think of it)?
(2) It $\chi : G \to k$ is the character of a matrix representation R of G, what is the character $\chi \downarrow_H : H \to k$ of $R \downarrow_H$?
(3) Show that if V is an irreducible G-module, then $V \downarrow_H$ is not necessarily an irreducible H-module.

Problem 10: (Optional - not to be handed in.) **Induction of representations.** Let G be a finite group and let $G \subseteq H$ be a subgroup of index $[G : H] = r$. Let $R : H \to GL_n(k)$ be a representation of H. We define a representation $R \uparrow^G : G \to GL_{nr}(k)$ (the *induction of R to G*) as follows. Let $T = \{t_1, t_2, \ldots, t_r\}$ be a left transversal for H in G, so that every element $g \in G$ may be written uniquely as $g = t_i h$ for some $1 \leq i \leq r$ and $h \in H$. Given $g \in G$, we define $R \uparrow^G (g)$ to be the block matrix

$$R \uparrow^G (g) := \begin{pmatrix} R(t_1^{-1} g t_1) & \cdots & R(t_r^{-1} g t_1) \\ & \vdots & \\ R(t_r^{-1} g t_1) & \cdots & R(t_r^{-1} g t_r) \end{pmatrix},$$

where we interpret $R(t_i^{-1} g t_j) = 0$ if $t_i^{-1} g t_j \notin H$.

(1) Prove that for any $g \in G$, the matrix $R \uparrow^G (g)$ is a 'block permutation matrix' – just one of the blocks in every row and column above is nonzero. Also show that $R \uparrow^G (1)$ is the identity matrix in $GL_{nr}(k)$.
(2) Prove that for any $x, y \in G$ we have $R \uparrow^G (xy) = R \uparrow^G (x) R \uparrow^G (y)$, so that $R \uparrow^G$ is actually a representation of G.
(3) Let $\chi : H \to k$ be the character of R. Prove that the character $\chi \uparrow^G$ of $R \uparrow^G$ is given by the formula

$$\chi \uparrow^G (g) = \sum_{i=1}^{r} \chi(t_i^{-1} g t_i),$$

where we make the convention that $\chi(x) = 0$ if $x \notin H$.
(4) Now assume $k = \mathbb{C}$. Prove that $\chi \uparrow^G$ is also given by the formula

$$\chi \uparrow^G (g) = \frac{1}{|H|} \sum_{x \in G} \chi(x g x^{-1}).$$

Deduce that the representation $R \uparrow^G$ does not depend on the choice of the transversal T, up to isomorphism.
(5) Give an example of an irreducible matrix representation R of H over $\mathbb{C}$ such that $R \uparrow^G$ is not irreducible.