Math 100C — Lecture 12 — 5/1/2017

§15.1 Fields

Recall, a <u>field</u> k is a commutative ring (w/ 1) such that every element of $k-\{0\}$ is a unit.

<u>Key Examples</u>

1. A <u>number field</u> is a subfield of $\mathbb{C}$ (eg $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$)
2. A <u>finite field</u> (e.g. $\mathbb{F}_p$, p prime $\mathbb{F}_2[x]/\langle x^2+x+1\rangle$.)
3. A field containing $\mathbb{C}(t)$ is a <u>function field</u>.

<u>Def</u> Let F be a field. An <u>extension</u> K of F is a field with $F \subseteq K$. Also write K/F or $\begin{array}{c} K \\ | \\ F \end{array}$ to mean K is an extension of F.

<u>Ex</u> $\mathbb{C}$ is an extension of $\mathbb{R}$, $\begin{array}{c} \mathbb{C} \\ | \\ \mathbb{R} \\ | \\ \mathbb{Q} \end{array}$.

§15.2 Algebraic and Transcendental Elements

<u>Def</u> Let K/F be a field extension and $\alpha \in K$. α is <u>algebraic over F</u> if $\exists$ a monic polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in F[x] \text{ s.t.}$$

$f(\alpha) = 0$. Otherwise, α is <u>transcendental over F</u>.

<u>Def</u> Let K/F be a field extension & let $\alpha_1, \ldots, \alpha_k \in K$.

$F[\alpha_1, \ldots, \alpha_k] :=$ smallest <u>subring</u> of K containing $\alpha_1, \ldots, \alpha_k$.

$F(\alpha_1, \ldots, \alpha_k) :=$ smallest <u>subfield</u> of K containing $\alpha_1, \ldots, \alpha_k$.

Prop Let K/F be an ~~alge~~ extension of fields & let $\alpha \in K$ be algebraic over F. Let $f(x) \in F[x]$ be a monic polynomial. TFAE:

① $\langle f(x) \rangle = \ker \varphi$, where $\varphi : F[x] \longrightarrow K$, $g(x) \longmapsto g(\alpha)$ is the evaluation map.

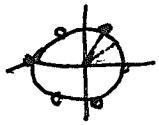
② $f(x)$ is the monic poly. of lowest degree w/ α as a root.

③ $f(x)$ is an irreducible elt of $F[x]$, and $f(\alpha) = 0$.

④ $f(\alpha) = 0$ and $\langle f(x) \rangle \subseteq F[x]$ is a maximal ideal.

⑤ $f(\alpha) = 0$ and, if $g(\alpha) = 0$ for some $g(x) \in F[x]$, then $f \mid g$.

Ex Consider $\mathbb{C}/\mathbb{Q}$ and let $\zeta = e^{\frac{2\pi i}{6}} \in \mathbb{C}$.



Then $\zeta^6 - 1 = 0$, so ζ is a root of $x^6 - 1 \in \mathbb{Q}[x]$ & ζ is <u>algebraic</u> over $\mathbb{Q}$.

However, $x^6 - 1$ is <u>NOT</u> the ~~minimal~~ irreducible poly. of ζ over $\mathbb{Q}$. We have

$$f(x) = (x - \zeta) \cdot (x - \zeta^{-1}) = x^2 - (\zeta + \zeta^{-1})x + (\zeta\zeta^{-1})$$
$$= x^2 - x + 1 \in \mathbb{Q}[x]$$

Then $f(\zeta) = 0$. Since f is <u>irreducible</u> in $\mathbb{Q}[x]$, f <u>is</u> the irred. polynomial for ζ over $\mathbb{Q}$.

Ex ① Consider $\mathbb{R}/\mathbb{Q}$ & let $\alpha = \sqrt[3]{3-\sqrt{4}} \in \mathbb{R}$.

Then $\alpha^3 = 3-\sqrt{4}$, $\alpha^6 = 13 - 6\sqrt{4}$ so that

$\alpha^6 - 6\alpha^3 + 5 = 0$ & α is a root of

$f(x) = x^6 - 6x^3 + 5 = 0$. $\Rightarrow$ α is <u>algebraic over $\mathbb{Q}$</u>.

② Consider $\mathbb{C}/\mathbb{Q}$ and let $\beta = e \cdot i \in \mathbb{C}$.

Then β is <u>transcendental/$\mathbb{Q}$</u>.

However, $\beta^2 = -e^2$, so β is a root of

$f(\beta) = 0$ if $f(x) = x^2 + e^2 \in \mathbb{R}[x]$ & β is

<u>algebraic/$\mathbb{R}$</u>.

If K/F is an extension of fields & $\alpha \in K$, we have an

<u>evaluation</u> <u>homomorphism</u> $\varphi : F[x] \longrightarrow K$

$g(x) \longmapsto g(\alpha)$.

with $\text{Image}\,\varphi = F[\alpha] = \{a_n\alpha^n + \cdots + a_1\alpha + a_0 : a_i \in F\} \subseteq K$.

If α is <u>transcendental</u>, $\ker\varphi = 0$ & $F[x] \cong F[\alpha]$.

(eg $\mathbb{Q}[\pi] \cong \mathbb{Q}[x]$.)

If α is <u>algebraic</u>, $\ker\varphi \neq 0$. Since $F[x]$

is a P.I.D., $\exists!$ monic poly. $f(x) \in F[x]$ s.t.

$\ker\varphi = \langle f(x) \rangle$.

Suppose $\alpha \in K$ is algebraic & K/F is an extension

<u>Def</u> The <u>irreducible</u> <u>poly</u> <u>for</u> α <u>over</u> F is the

monic polynomial $f(x) \in F[x]$ st $\ker\varphi = \langle f(x) \rangle$, where $\varphi : F[x] \to K$, $g(x) \mapsto g(\alpha)$.