§ 15.1 Fields
Recall, a field \( k \) is a commutative ring (w/ 1)
such that every element of \( k-\{0\} \) is a unit.

Key Examples

1. A number field is a subfield of \( \mathbb{C} \) (e.g. \( \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots \))
2. A finite field (e.g. \( \mathbb{F}_p \), \( p \) prime \( \mathbb{F}_2[y]/\langle y^2+y+1 \rangle \).)
3. A field containing \( \mathbb{C}(t) \) is a function field.

Def Let \( F \) be a field. An extension \( K \) of \( F \)
is a field with \( F \subseteq K \). Also write \( K/F \)
\( K \) to mean \( K \) is an extension of \( F \).

Ex \( \mathbb{C} \) is an extension of \( \mathbb{R} \),
\( \mathbb{R} \) is an extension of \( \mathbb{Q} \).

§ 15.2 Algebraic and Transcendental Elements

Def Let \( K/F \) be a field extension and \( \alpha \in K \).
\( \alpha \) is algebraic over \( F \) if \( \exists \) a monic
polynomial
\[ f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \in F[x] \]
s.t. \( f(\alpha) = 0 \). Otherwise, \( \alpha \) is transcendental over \( F \).
Def Let $K/F$ be a field extension & let $\alpha_1, \ldots, \alpha_k \in K$.

$F[\alpha_1, \ldots, \alpha_k] := \text{smallest subring of } K \text{ containing } \alpha_1, \ldots, \alpha_k$.

$F(\alpha_1, \ldots, \alpha_k) := \text{smallest subfield of } K \text{ containing } \alpha_1, \ldots, \alpha_k$. 
Prop. Let $K/F$ be an algebraic extension of fields & let $\alpha \in K$ be algebraic over $F$. Let $f(x) \in F[x]$ be a monic polynomial. TFAE:

1. $\langle f(x) \rangle = \ker \varphi$, where $\varphi : F[x] \to K$ is the evaluation map,
2. $f(x)$ is the monic poly. of lowest degree w/ $\alpha$ as a root,
3. $f(x)$ is an irreducible elt of $F[x]$, and $f(\alpha) = 0$,
4. $f(\alpha) = 0$ and $\langle f(x) \rangle \subseteq F[x]$ is a maximal ideal,
5. $f(\alpha) = 0$ and if $g(x) = 0$ for some $g(x) \in F[x]$, then $f \mid g$.

Ex. Consider $\mathbb{C}/\mathbb{Q}$ and let $\zeta = e^{\frac{2\pi i}{6}} \in \mathbb{C}$.

Then $\zeta^6 - 1 = 0$, so $\zeta$ is a root of $x^6 - 1 \in \mathbb{Q}[x]$ & $\zeta$ is algebraic over $\mathbb{Q}$.

However, $x^6 - 1$ is not the irreducible poly. of $\zeta$ over $\mathbb{Q}$. We have

$$f(x) = (x - \zeta)(x - \zeta^{-1}) = x^2 - (\zeta + \zeta^{-1})x + (\zeta \cdot \zeta^{-1}) = x^2 - x + 1 \in \mathbb{Q}[x]$$

Then $f(\zeta) = 0$. Since $f$ is irreducible in $\mathbb{Q}[x]$, $f$ is the irred. polynomial for $\zeta$ over $\mathbb{Q}$.
Example 1: Consider \( \mathbb{R}/\mathbb{Q} \) and let \( \alpha = \sqrt[3]{3} \sqrt[4]{4} \in \mathbb{R} \).

Then \( \alpha^3 = 3 \sqrt[4]{4} \), \( \alpha^6 = 13 - 6\sqrt[4]{4} \) so that
\( \alpha^6 - 6\alpha^3 + 5 = 0 \) and \( \alpha \) is a root of
\[ f(x) = x^6 - 6x^3 + 5 = 0. \Rightarrow \alpha \text{ is algebraic over } \mathbb{Q}. \]

2. Consider \( \mathbb{C}/\mathbb{Q} \) and let \( \beta = e^{i\pi} \in \mathbb{C} \).

Then \( \beta \) is transcendental over \( \mathbb{Q} \).

However, \( \beta^2 = -1 \), so \( \beta \) is a root of
\[ f(\beta) = 0 \text{ if } f(x) = x^2 + 1 \in \mathbb{R}[x] \text{ and } \beta \text{ is algebraic over } \mathbb{R}. \]

If \( K/F \) is an extension of fields and \( \alpha \in K \), we have an evaluation homomorphism \( \varphi : F[x] \to K \)
\[ \varphi(x) \mapsto \varphi(\alpha). \]

with image \( \varphi = F[x] = \{ a_n x^n + \ldots + a_1 x + a_0 : a_i \in F \} \subseteq K. \)

If \( \alpha \) is transcendental, \( \ker \varphi = 0 \) and \( F[x] \cong F[\alpha]. \)
(eg \( \mathbb{Q}[x] \cong \mathbb{Q}[\alpha] \).

If \( \alpha \) is algebraic, \( \ker \varphi \neq 0 \). Since \( F[x] \)
is a P.I.D., \( \exists! \) monic poly. \( f(x) \in F[x] \) s.t.
\[ \ker \varphi = \langle f(x) \rangle. \]

Suppose \( \alpha \in K \) is algebraic and \( K/F \) is an extension.

Define the irreducible poly for \( \alpha \) over \( F \) is the monic polynomial \( f(x) \in F[x] \) s.t. \( \ker \varphi = \langle f(x) \rangle \), where \( g(x) \in g(x) \).