Math 100C: Spring 2017
Practice Final Exam

Instructions: This is a 3 hour exam. No books, notes, electronic devices, or interpersonal assistance are allowed. Write your answers in your blue book, making it clear what problem you are working on. Be sure to justify your answers. This exam is out of 100 points; you get 5 points for legibly writing your name on your blue book. Good luck!

Problem 1: [15 pts.] Let G be a finite group and let $R : G \to GL_n(\mathbb{C})$ be a complex matrix representation of G. For $g \in G$, prove that R_g is conjugate to a diagonal matrix $\mathrm{diag}(\omega_1, \ldots, \omega_n)$ where $\omega_i^k = 1$ for $1 \leq i \leq n$.

Problem 2: [15 pts.] Write down the character table of the dihedral group D_6 of symmetries of a regular hexagon.

Problem 3: [5 + 10 pts.] Let V be a representation of a finite group G over $\mathbb{C}$ and let $\mathrm{End}(V)$ be the space of all $\mathbb{C}$-linear maps $V \to V$.

(a) Prove the map $\Phi : \mathrm{End}(V) \to \mathrm{End}_G(V)$ given by

$$\Phi(T)(v) = \frac{1}{|G|} \sum_{g \in G} g.T(g^{-1}.v)$$

is a well-defined linear projection.

(b) If V is irreducible and $T \in \mathrm{End}(V)$, prove that there is a complex number λ such that $\Phi(T)(v) = \lambda v$ for all $v \in V$.

Problem 4: [5 + 10 pts.] Let K/F be an extension of fields.

(a) Carefully state what it means for K/F to be 'algebraic'.

(b) Give an example of an algebraic extension which is not finite.

Problem 5: [10 pts.] Let $F \subseteq L \subseteq K$ be a tower of fields with K/F finite such that both K/L and L/F are Galois. Prove that K/F is not necessarily Galois.

Problem 6: [10 pts.] Give an explicit construction of the field with 25 elements as a quotient of the polynomial ring $\mathbb{F}_5[x]$.

Problem 7: [5 + 5 + 10 pts.] Let $\alpha = 3^{1/4}$ be the real fourth root of 3 and let $K = \mathbb{Q}(\alpha, i)$.

(a) Prove that $K/\mathbb{Q}$ is Galois.

(b) Determine the Galois group $G(K/\mathbb{Q})$.

(c) Find all intermediate fields $\mathbb{Q} \subseteq L \subseteq K$ with $L/\mathbb{Q}$ Galois.