1. Let $J \in \text{Mat}_n(\mathbb{C})$ be the Jordan canonical form of $R_g$. If $B = \begin{pmatrix} \lambda & 1 \\ \frac{1}{\lambda} & \lambda \\ \vdots & \ddots & \ddots & \ddots \\ \frac{1}{\lambda^{m-1}} & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \end{pmatrix}$ is a typical Jordan block of $J$, a straightforward induction shows $B^m$ has the form

$$B^m = \begin{pmatrix} \lambda^m & m \lambda^{m-1} & \ldots & \lambda \\ 0 & \lambda^m & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \lambda^m \\ \end{pmatrix}$$

for all $m > 1$.

Since $G$ is finite, $g \in G$ has finite order $k$, let $k$ be the order of $g$ in $G$. Then $(R_g)^k = R_{g^k} = R_1 = I$, so $J^k = I$. By $\bigstar$, this implies that every Jordan block of $J$ has size 1:

$$J = \text{diag}(\omega_1, \ldots, \omega_n), \quad \omega_i \in \mathbb{C}.$$ Since $J^k = I$, we have $\omega_i^k = 1$ for $1 \leq i \leq n$. \hfill \square
We have $D_6 = \langle r, s | r^6 = s^2 = 1, srs = r^{-1} \rangle$.

The conjugacy classes are...

\[
\begin{align*}
\{1\}, \{r^3, r^{-3}\}, \{s, r^2, sr^2, sr^4\}, \{sr, sr^3, sr^5\},
\end{align*}
\]

so we need to find 6 irreducible chars $\chi_1 \cdots \chi_6 : D_6 \rightarrow \mathbb{C}^*$.

We start by finding the 1-dim'l irreducible characters; these are $\chi : D_6 \rightarrow \mathbb{C}^*$ satisfying

\[
\chi(r)^6 = 1, \chi(s)^2 = 1, \chi(s)\chi(r)\chi(s) = \chi(r)^2.
\]

The solns to these eqns are $\chi_1 : r \mapsto 1, s \mapsto 1, \chi_2 : r \mapsto 1, s \mapsto -1, \chi_3 : r \mapsto -1, s \mapsto 1, \chi_4 : r \mapsto -1, s \mapsto -1,$ giving the first 4 rows of our table.

$\chi_5$ is the character of the "defining" 2-dim'l rep'n of $D_6$ on $\mathbb{C}^2$. The dimn $n_6$ of $\chi_6$ is determined by the Magic Formula: $|D_6| = 12 = 1 + 1 + 1 + 1 + 2 + 2 + \ldots + 2 + n_6$, so $n_6 = 2$. $\chi_6$ is determined from character orthogonality.

<table>
<thead>
<tr>
<th>$\chi_1$</th>
<th>$\chi_2$</th>
<th>$\chi_3$</th>
<th>$\chi_4$</th>
<th>$\chi_5$</th>
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</tbody>
</table>
(3) (a) Let \( h \in G \) and \( v \in V \). Then for \( T \in \text{End}_G(V) \),

\[
\Phi(T)(h,v) = \frac{1}{|G|} \sum_{g \in G} g \cdot T(g^{-1}h,v)
\]

\[
x^{-1} = g^{-1}h \quad \Rightarrow \quad x^0 = h \cdot g \quad \rightarrow \quad \frac{1}{|G|} \sum_{x \in G} h \cdot x^0 \cdot T(x^{-1},v)
\]

\[
= h \cdot \left( \frac{1}{|G|} \sum_{x \in G} x \cdot T(x^0,v) \right) = h \cdot \Phi(T)(v),
\]

so \( \Phi: \text{End}(V) \rightarrow \text{End}_G(V) \) is well-defined.

Clearly \( \Phi \) is linear. If \( T \in \text{End}_G(V) \) then

\[
\Phi(T)(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot T(g^{-1},v) = \frac{1}{|G|} \sum_{g \in G} g \cdot g^{-1} \cdot T(v)
\]

\[
= T(v),
\]

so \( \Phi(T) = T \) and \( \Phi \) is a projection.

(b) By (a), we have \( \Phi(T) \in \text{End}_G(V) \). By Schur's Lemma, since \( V \) is fin-diml \( / \mathbb{C} \),

\[
\exists \lambda \in \mathbb{C} \text{ s.t. } \Phi(T) = \lambda \cdot \text{id}_V.
\]

(4) (a) \( K/F \) is algebraic if \( \forall \alpha \in K \) \exists a monic polynomial \( f(x) \in F[x] \) s.t. \( f(\alpha) = 0 \).

(b) Let \( F = \mathbb{Q} \) and \( K = \mathbb{Q}(\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}, \ldots) \).

Since \( x^n - 2 \in \mathbb{Q}[x] \) is \( 2 \)-Eisenstein \& irreducible \( / \mathbb{Q} \),

the degree of \( n\sqrt{2} \) over \( \mathbb{Q} \) is \( n \) for all \( n \geq 2 \).

Thus \( [K:F] \geq [\mathbb{Q}(n\sqrt{2}) : \mathbb{Q}] = n \), \( \forall n \geq 2 \), so
K/F is not finite. If \( \alpha \in K \), \( \exists N > 2 \) s.t.

\[
\alpha \in \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \ldots, \sqrt[N]{2}).
\]

But

\[
[\mathbb{Q}(\alpha) : \mathbb{Q}] 
\leq [\mathbb{Q}(\sqrt{2}, \ldots, \sqrt[N]{2}) : \mathbb{Q}]
\leq [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] \cdots [\mathbb{Q}(\sqrt[N]{2}) : \mathbb{Q}]
\]

\[
= 2 \cdot 3 \cdots N < \infty,
\]

so \( \mathbb{Q}(\alpha) / \mathbb{Q} \) is finite & thus \( \alpha \) is algebraic over \( \mathbb{Q} \).

(5) Let \( \alpha = 2^{\frac{1}{2}} \) and \( \beta = 2^{\frac{1}{4}} \), \( \alpha, \beta \in \mathbb{R} > 0 \).

Let \( F = \mathbb{Q} \), \( L = \mathbb{Q}(\alpha) \), and \( K = \mathbb{Q}(\beta) \), so \( F \subseteq L \subseteq K \). Since \( L \) is the splitting field of \( x^2 - 2 \in F[x] \) over \( F \), \( L/F \) is Galois.

Since \( K \) is the splitting field of \( x^2 - \alpha \in L[x] \) over \( L \), \( K/L \) is Galois. However, \( K/F \) is not Galois: since \( f(x) = x^4 - 2 \) is irreducible \( \mathbb{Q} \) (2-Eisenstein) & has \( \beta \) as a root, \( [K:F] = 4 \). However, since \( f(\alpha) \in F[x] \), and any \( \sigma \in G(K/F) \) is determined by \( \sigma(\beta) \), and must have \( \sigma(\beta) \in \{ \text{roots of } f(x) \} = \{ \pm \beta, \pm i \beta \} \).

Since \( \mathbb{Q}(\beta) = K \subseteq \mathbb{R} \), this means \( \sigma(\beta) \in \{ \pm \beta \} \), so \( |G(K/F)| \leq 2 \). Thus \( |G(K/F)| < [K:F] \) so \( K/F \) is not Galois.
satisfies \( \sigma(\alpha) \in \{ \alpha, i\alpha, -\alpha, -i\alpha \} \) and
\( \sigma(i) \in \{ i, -i \} \).

This gives \( \leq 8 \) choices for \( \sigma \in G(\mathbb{Q}(\sqrt{i})/\mathbb{Q}) = G \), since \( |G| = 8 \), all choices are possible. Define \( \sigma, \tau \in G \) by
\( \sigma : \alpha \mapsto i\alpha \), \( \tau : i \mapsto i \). Then \( \sigma^2 = \tau^2 = 1 \).

Also, \( \tau \sigma \tau(\alpha) = \tau \sigma(\alpha) = \tau(i\alpha) = -i\alpha \) and
\( \tau \sigma \tau(i) = \tau \sigma(-i) = \tau(-i) = i \) so
\( \tau \sigma \tau = \sigma^{-1} \). Thus \( G = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = 1, \tau \sigma \tau = \sigma^{-1} \rangle \)
is isom. to \( D_4 \).

(c) The normal subgroups of \( D_4 = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = 1, \tau \sigma \tau = \sigma^{-1} \rangle \)
are \( \langle \sigma \rangle, \langle \tau \rangle, \langle \tau, \sigma^2 \tau \rangle, \langle \sigma \tau \rangle, \langle \sigma^3 \tau \rangle, \langle \sigma^2 \rangle, \langle \tau^2 \rangle \), \( 1 \).

By the Fundamental Thm. of Galois Theory, our intermediate extensions \( w/ L/\mathbb{Q} \) Galois are \( L = K^N \), where \( N \leq G(\mathbb{Q}(\sqrt{i})) \)
is a normal subgroup. We compute:

\[
\frac{N}{\langle \sigma, \tau \rangle} \quad \frac{\mathbb{Q}}{\langle \sigma \rangle} \quad \frac{\mathbb{Q}(i)}{\langle \sigma \rangle} \quad \text{(since deg}_{\mathbb{Q}} i = 2)} \quad \frac{[D_4 : \langle \sigma \rangle]}{[D_4 : \langle \tau, \sigma^2 \tau \rangle]}
\]
\[
\frac{\mathbb{Q}(\alpha^2)}{\langle \sigma \rangle} \quad \text{(since deg}_{\mathbb{Q}} \alpha^2 = \text{deg}_{\mathbb{Q}} \sqrt{3} = 2)} \quad \frac{[D_4 : \langle \tau, \sigma^2 \tau \rangle]}{[D_4 : \langle \sigma \tau, \sigma^3 \tau \rangle]}
\]
\[
\frac{\mathbb{Q}(i, \alpha^2)}{\langle \sigma \rangle} \quad \text{(since deg}_{\mathbb{Q}} \text{i} \alpha^2 = \text{deg}_{\mathbb{Q}} \sqrt{3} = 2 = [D_4 : \langle \sigma \tau, \sigma^2 \tau \rangle]} \quad \frac{[\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}] = 4}{[D_4 : \langle \sigma^2 \rangle]}
\]
\[
1 \quad K \quad \mathbb{Q}
\]
(6) We find a monic irreducible quadratic polynomial \( f(x) \in F_5[x] \). Take

\[ f(x) = x^2 + x + 1. \]

Then (in \( F_5 \))

\[ f(0) = 1, \quad f(1) = 3, \quad f(2) = 7 = 2, \quad f(3) = 13 = 3, \quad f(4) = 21 = 1, \]

so \( f(x) \) has no roots in \( F_5 \). Since \( \deg f = 2 \),

\( f \) is irreducible. \( \overline{F_5[x]} / \langle f(x) \rangle \) is a field with \( 25 = 5^2 \) elements.

(7) (a) Let \( f(x) = x^4 - 3 \in \mathbb{Q}[x] \), so \( f \) has roots \( \pm \alpha, \pm i \alpha \). The splitting field for \( f/\mathbb{Q} \) is

\[ \mathbb{Q}(\pm \alpha, \pm i \alpha) = \mathbb{Q}(\alpha, i) = K. \]

Thus \( K \) is the splitting field for \( f(x)/\mathbb{Q} \), so \( K/\mathbb{Q} \) is Galois.

(b) Since \( f(\alpha) \) is 3-Eisensteinian, \( f(x) \) is irreducible. Also \[ [\mathbb{Q}(\alpha) : \mathbb{Q}] = \deg f = 4. \]

Also \[ [\mathbb{Q}(i) : \mathbb{Q}] = 2. \]

Since \( \mathbb{Q}(\alpha) \subseteq \mathbb{R} \), \( i \notin \mathbb{Q}(\alpha) \) and \[ [\mathbb{Q}(\alpha, i) : \mathbb{Q(\alpha)}] = 2, \]

so \[ [K : \mathbb{Q}] = [\mathbb{Q}(\alpha, i) : \mathbb{Q(\alpha)}] \cdot [\mathbb{Q}(\alpha) : \mathbb{Q}] = 2 \cdot 4 = 8. \]

Since \( K/\mathbb{Q} \) is Galois, \( G := G(K/\mathbb{Q}) \) has \( |G| = 8 \). Let \( g(x) = x^2 + 1 \in \mathbb{Q}[x] \). Then \( G \) acts on the roots of \( f(\alpha) \) and \( g(\alpha) \), so that \( \sigma \in G \).