Problem 1: [5 + 15 pts.] Let $G$ be a group and let $V$ be a nonzero $G$-module over some field $k$.
(a) Carefully state what it means for $V$ to be “indecomposable”.
(b) Suppose $V$ is irreducible. Prove that $V$ is indecomposable.

Problem 2: [25 pts.] Let $G$ be a finite group and let $V$ be a finite-dimensional $G$-module over $\mathbb{C}$. The Reynolds operator $R : V \to V$ is defined by

$$R(v) := \frac{1}{|G|} \sum_{g \in G} g.v,$$

for all $v \in V$. Prove that $R$ projects $V$ onto the subspace $V^G \subseteq V$ of $G$-invariants:

$$V^G := \{v \in V : g.v = v \text{ for all } g \in G\}.$$

Problem 3: [25 pts.] Write down the character table of the product group $S_3 \times S_2$.

Problem 4: [25 pts.] Consider the additive group of integers $G = \mathbb{Z}$. Give an example of two complex matrix representations

$$R, S : \mathbb{Z} \to GL_2(\mathbb{C})$$

of $\mathbb{Z}$ of dimension 2 such that we have an equality of characters

$$\chi_R = \chi_S,$$

but such that $R$ and $S$ are not isomorphic.