Math 103A: Winter 2014
Homework 6
Due 5:00pm on Friday 2/21/2014

Problem 1: (Exercise 7.8 in Gallian) Suppose that \( a \) has order 15. Find all of the left cosets of \( \langle a^5 \rangle \) in \( \langle a \rangle \).

Problem 2: (Exercise 7.10 in Gallian) Find an example of a group \( G \) and subgroups \( H \) and \( K \) such that \( HK = \{hk : h \in H, k \in K\} \) is not a subgroup of \( G \).

Problem 3: Suppose \( H \) and \( K \) are subgroups of a group \( G \). If \( |H| = 12 \) and \( |K| = 35 \), what is \( |H \cap K| \)?

Problem 4: Let \( G = GL(n, \mathbb{R}) \) and let \( H = SL(n, \mathbb{R}) \). Let \( A \) be an \( n \times n \) real matrix such that \( \det(A) = 3 \). Prove that
\[
AH = \{ B \in \text{Mat}_{n \times n}(\mathbb{R}) : \det(B) = 3 \}.
\]

Problem 5: (Exercise 7.40 in Gallian) Prove that any group of order 63 contains an element of order 3.

Problem 6: (Exercise 7.64 in Gallian) A soccer ball has 20 faces that are regular hexagons and 12 faces that are regular pentagons. Use the Orbit-Stabilizer Theorem to show that a soccer ball cannot have a 60° rotational symmetry about a line through the centers of two opposite hexagonal faces. (Hint: You may use without proof the fact that the group of rotational symmetries of a soccer ball has order 60 – and is in fact isomorphic to \( A_5 \)).

Problem 7: (Exercise 8.12 in Gallian) Give examples of four groups of order 12, no two of which are isomorphic. Give reasons why no two are isomorphic.

Problem 8: (Exercise 8.16 in Gallian) Suppose that \( G_1 \approx G_2 \) and \( H_1 \approx H_2 \). Prove that \( G_1 \oplus H_1 \approx G_2 \oplus H_2 \).

Problem 9: (Exercise 8.30 in Gallian) Find all subgroups of order 4 in \( Z_4 \oplus Z_4 \).