**Problem 1:** Let $x \geq -1$ be a real number and let $n$ be a nonnegative integer. Prove Bernoulli’s Inequality:

$$(1 + x)^n \geq 1 + nx.$$  

**Solution:** Fix a real number $x \geq -1$. We induct on $n$.

If $n = 0$ then $(1 + x)^n = (1 + x)^0 = 1$ and $1 + nx = 1 + 0(x) = 1$, so Bernoulli’s Inequality is true in this case.

Fix a nonnegative integer $n$ and inductively assume that $(1 + x)^n \geq 1 + nx$. We have that

$$(1 + x)^{n+1} = (1 + x)^n (1 + x) \geq (1 + nx)(1 + x) = 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x,$$

where the first inequality used the inductive hypothesis and the second inequality used the fact that $nx^2 \geq 0$.

By induction, we conclude that $(1 + x)^n \geq 1 + nx$ for all nonnegative integers $n$, as desired.

**Problem 2:** Prove that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

for all positive integers $n$.

**Solution:** We induct on $n$. If $n = 1$ then

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{2} = \frac{1}{n+1},$$

so the claimed equality is true in this case.

Fix a positive integer $n$ and inductively assume that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

We have

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{1}{(n+1)(n+2)} + \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{(n+1)(n+2)} + \frac{n}{n+1} = \frac{n+1}{n+2},$$

where the second equality used the inductive hypothesis.

By induction, we conclude that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all positive integers $n$, as desired.
Problem 3: What is wrong with the following ‘proof’ that all cupcakes have the same flavor?

“Suppose there are $n$ cupcakes in the world. We induct on $n$. If $n = 1$ the result is clear (since any cupcake has the same flavor as itself).

Fix a positive integer $n > 1$ and inductively assume that any collection of $n - 1$ cupcakes has the same flavor. We show that all $n$ of the cupcakes in the world have the same flavor. To do this, place the cupcakes in a line and label them $c_1, c_2, \ldots, c_n$. By the inductive hypothesis, the first $n - 1$ cupcakes $c_1, c_2, \ldots, c_{n-1}$ and the last $n - 1$ cupcakes $c_2, c_3, \ldots, c_n$ all have the same flavor. In particular, the cupcake $c_1$ has the same flavor as the middle cupcakes $c_2, c_3, \ldots, c_{n-1}$. Also, the cupcake $c_n$ has the same flavor as the middle cupcakes $c_2, c_3, \ldots, c_{n-1}$. We conclude that all of the cupcakes $c_1, c_2, \ldots, c_{n-1}, c_n$ have the same flavor, as desired.”

Solution: The flaw here lies in the inductive step. Consider the case where $n = 2$. In this case, we have two cupcakes, $c_1$ and $c_2$. In this case, there are no middle cupcakes. It is true that $c_1$ has the same flavor of all of the (nonexistent) middle cupcakes and $c_2$ has the same flavor as all of the (nonexistent) middle cupcakes. However, since there are no middle cupcakes, we cannot conclude that $c_1$ and $c_2$ have the same flavor.

Problem 4: What is wrong with the following ‘proof’ that 1 is the largest integer? What does this argument actually show?

“Let $n$ be the largest integer. Since 1 is an integer we must have $1 \leq n$. Since $n^2$ is also an integer we must have $n^2 \leq n$. Since $n$ is positive, the inequality $n^2 \leq n$ implies $n \leq 1$. Since $1 \leq n$ and $n \leq 1$, we have that $n = 1$, as desired.”

Solution: In the first sentence of the argument, it is implicitly assumed that a largest integer $n$ exists. Given this, the argument provides a logically valid demonstration that such a largest integer must equal 1. The argument therefore proves by contradiction that there is no largest integer.

Problem 5: Let $A, B, C$ be sets. Prove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Solution: Let $x$ be an object. We have

$$x \in A \cup (B \cap C) \iff x \in A \text{ or } x \in B \cap C$$
$$\iff x \in A \text{ or } (x \in B \text{ and } x \in C)$$
$$\iff (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$$
$$\iff (x \in A \cup B) \text{ and } (x \in A \cup C)$$
$$\iff x \in (A \cup B) \cap (A \cup C).$$
Problem 6: Given a real number \(a\), we define the half-infinite intervals \([a, \infty)\) and \((a, \infty)\) by

\[
[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}, \\
(a, \infty) = \{x \in \mathbb{R} \mid x > a\}.
\]

Prove that following.

1. \([a, \infty) \subset (b, \infty)\) if and only if \(a > b\).
2. \((a, \infty) \subset [b, \infty)\) if and only if \(a \geq b\).

Solution:

1. Suppose \([a, \infty) \subset (b, \infty)\). Then \(a \in [a, \infty)\), so that \(a \in (b, \infty)\). This implies \(a > b\).

   Now suppose \(a > b\). Let \(x \in [a, \infty)\). Then \(x \geq a > b\), so that \(x \in (b, \infty)\). We conclude that \([a, \infty) \subseteq (b, \infty)\). Observe that \(a > \frac{a+b}{2} > b\), so that \(\frac{a+b}{2} \in (b, \infty)\). It follows that \([a, \infty) \subseteq (b, \infty)\).

2. Suppose \((a, \infty) \subset [b, \infty)\). If \(a < b\), then we could have \(a < \frac{a+b}{2} < b\), so that \(\frac{a+b}{2} \in (a, \infty) - [b, \infty) = \emptyset\). This contradiction forces \(a \geq b\).

   Suppose \(a \geq b\). Let \(x \in (a, \infty)\). Then \(x > a \geq b\), so \(x \in [b, \infty)\). We conclude that \((a, \infty) \subseteq [b, \infty)\). Moreover, since \(a \geq b\) we have \(b \in [b, \infty) - (a, \infty)\). We conclude that \((a, \infty) \subset [b, \infty)\).

Problem 7: Let \(f : \mathbb{R} \to \mathbb{R}\) be a function from the real numbers to itself. The function \(f\) is called uniformly continuous if for all \(\varepsilon > 0\), there exists \(\delta > 0\) such that for any pair of real numbers \(x, y\) with \(|x - y| < \delta\), we have that \(|f(x) - f(y)| < \varepsilon\).

What does it mean for a function \(f : \mathbb{R} \to \mathbb{R}\) to not be uniformly continuous? Observe that you do not need to ‘know’ anything about uniform continuity to answer this question.

Solution: \(f\) is not uniformly continuous if there exists \(\varepsilon > 0\) such that for any \(\delta > 0\) there exists a pair of real numbers \(x, y\) with \(|x - y| < \delta\) such that \(|f(x) - f(y)| \geq \varepsilon\).

Problem 8: Prove or give a counterexample to each of the following statements.

1. \(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0\).
2. \(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x - y > 0\).
3. \(\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y > 0\).
4. \(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy > 0\).
5. \(\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy > 0\).
6. \(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy \geq 0\).
7. \(\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy \geq 0\).
8. \(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (x + y > 0 \text{ or } x + y = 0)\).
9. \(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (x + y > 0 \text{ and } x + y = 0)\).
10. \((\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0)\) and \((\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 0)\).

Solution:

1. This is true. Let \(x \in \mathbb{R}\). Take \(y = -x + 1\). Then \(x + y = 1 > 0\).
(2) This is true. Let $x \in \mathbb{R}$. Take $y = x - 1$. Then $x - y = 1 > 0$.

(3) This is false. Let $x \in \mathbb{R}$. If $y = -x$, then $x + y = 0$.

(4) This is false. Take $x = 0$. For any $y \in \mathbb{R}$ we have $xy = 0y = 0$.

(5) This is false. Let $x \in \mathbb{R}$. If we take $y = 0$, then $xy = 0x = 0$.

(6) This is true. Let $x \in \mathbb{R}$. Take $y = 0$. Then $xy = 0x = 0$.

(7) This is true. Take $y = 0$. For any $y \in \mathbb{R}$ we have $xy = 0y = 0$.

(8) This is true. Let $x \in \mathbb{R}$. Take $y = -x$. Then $x + y = 0$ is true, so $(x + y > 0$ or $x + y = 0)$ is also true.

(9) This is false. Let $x = 0$. For any $y \in \mathbb{R}$ we have that $x + y = y > 0$ and $x + y = y = 0$ cannot hold simultaneously.

(10) This is true. Let $x \in \mathbb{R}$ and take $y = -x + 1$. Then $x + y = 1 > 0$, so the statement in the first pair of parentheses is true. Let $x \in \mathbb{R}$ and take $y = -x$. Then $x + y = 0$, so the statement in the second set of parentheses is true. Therefore, the overall statement is true.