Problem 1: Let $f : A \to B$ and $g : C \to D$ be functions. Define a new function
$$f \times g : A \times C \to B \times D$$
by the formula $(f \times g)(a, c) = (f(a), g(c))$.

1. Prove or disprove: If $f$ and $g$ are injective, then $f \times g$ is injective.
2. Prove or disprove: If $f$ and $g$ are surjective, then $f \times g$ is surjective.

Solution:

1. This is true. Suppose $f$ and $g$ are injections. Let $(a, c), (a', c') \in A \times C$ and suppose $(f \times g)(a, c) = (f \times g)(a', c')$. This means $(f(a), g(c)) = (f(a'), g(c'))$, which is to say $f(a) = f(a')$ and $g(c) = g(c')$. Since $f$ and $g$ are injections, this implies $a = a'$ and $c = c'$. We conclude that $(a, c) = (a', c')$ so that $f \times g$ is an injection.

2. This is true. Suppose $f$ and $g$ are surjections. Let $(b, d) \in B \times D$. Since $f$ is a surjection, there exists $a \in A$ such that $f(a) = b$. Since $g$ is a surjection, there exists $c \in C$ such that $g(c) = d$. We conclude that $(f \times g)(a, c) = (f(a), g(c)) = (b, d)$, so that $f \times g$ is a surjection.

Problem 2: Let $X$ be a set. For any subset $A \subseteq X$, the characteristic function $\chi_A : X \to \{0, 1\}$ is defined by
$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Let $1 : X \to \{0, 1\}$ be the constant function $1(x) = 1$. Let $A, B \subseteq X$. Prove the following.

1. We have $\chi_A \cdot \chi_B = \chi_{A \cap B}$.
2. We have $1 - \chi_A = \chi_A^c$.
3. We have $1 - (1 - \chi_A) \cdot (1 - \chi_B) = \chi_{A \cup B}$.

Solution:

1. Let $x \in X$. We have that
$$\chi_A(x) \cdot \chi_B(x) = \begin{cases} 1 \cdot 1 & x \in A \cap B \\ 1 \cdot 0 & x \in A - B \\ 0 \cdot 1 & x \in B - A \\ 0 \cdot 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & x \in A \cap B \\ 0 & x \notin A \cap B \end{cases} = \chi_{A \cap B}(x).$$
Therefore, we have $\chi_A \cdot \chi_B = \chi_{A \cap B}$.

1The functions on the left sides of these equations are defined pointwise, i.e., $\chi_A \cdot \chi_B(x) = \chi_A(x) \cdot \chi_B(x)$ for any $x \in X$, etc.
(2) Let \( x \in X \). We have that
\[
1(x) - \chi_A(x) = \begin{cases} 
1 - 1 & x \in A \\
1 - 0 & x \in A^c
\end{cases} = \begin{cases} 
0 & x \in A_c \\
1 & x \in A^c = \chi_{A^c}(x).
\end{cases}
\]
We conclude that \( 1 - \chi_A = \chi_{A^c} \).

(3) Applying Parts (1) and (2), we have
\[
1 - (1 - \chi_A) \cdot (1 - \chi_B) = 1 - \chi_{A^c} \cdot \chi_{B^c}
\]
\[
= 1 - \chi_{(A^c \cap B^c)}
= \chi_{(A^c \cap B^c)^c}
= \chi_{A \cup B}.
\]

Problem 3: (You will thank me when you take Math 140 or 142; the intuition comes from the definition.)

Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. Given \( x \in \mathbb{R} \), \( f \) is said to be continuous at \( x \) if for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) so that for any \( y \in \mathbb{R} \) with \( |x - y| < \delta \), we have \( |f(x) - f(y)| < \varepsilon \). The function \( f \) is said to be continuous if it is continuous at every \( x \in \mathbb{R} \).

Define a function \( f : \mathbb{R} \to \mathbb{R} \) by
\[
f(x) = \begin{cases} 
0 & x \leq 0 \\
1 & x > 0.
\end{cases}
\]

Prove or disprove: the function \( f \) is continuous.

Solution: We claim that \( f \) is not continuous. To see this, it is enough to show that \( f \) is not continuous at \( 0 \). Indeed, take \( \varepsilon = \frac{1}{2} > 0 \). For any \( \delta > 0 \), we have that
\[
|f(\delta) - f(0)| = |1 - 0| = 1 \geq \frac{1}{2} = \varepsilon.
\]
We conclude that \( f \) is not continuous at \( 0 \), so that \( f \) is not continuous.

Problem 4: (See Problem 3. Again, you will thank me later.)

Let \( f, g : \mathbb{R} \to \mathbb{R} \) be continuous functions. Prove that \( f + g : \mathbb{R} \to \mathbb{R} \) is also continuous, where \( f + g(x) = f(x) + g(x) \).

Hint: You may assume without proof the triangle inequality. For any \( x, y, z \in \mathbb{R} \) we have
\[
|x - z| \leq |x - y| + |y - z|.
\]

Solution: Let \( x \in \mathbb{R} \). We prove that \( f + g \) is continuous at \( x \). To see this, let \( \varepsilon > 0 \). Since \( f \) is continuous at \( x \), there exists \( \delta_1 > 0 \) such that \( |x - y| < \delta_1 \) implies \( |f(x) - f(y)| < \frac{\varepsilon}{2} \). Since \( g \) is continuous at \( x \), there exists \( \delta_2 > 0 \) such that \( |x - y| < \delta_2 \)

implies $|g(x) - g(y)| < \frac{\varepsilon}{2}$. Let $\delta = \min(\delta_1, \delta_2)$. Then $\delta > 0$ and if $|x - y| < \delta$ we have

\[
|(f + g)(x) - (f + g)(y)| = |f(x) - f(y) + g(x) - g(y)| \\
\leq |f(x) - f(y)| + |g(x) - g(y)| \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

We conclude that $f + g$ is continuous at $x$. Since $x \in \mathbb{R}$ was arbitrary, we have that $f + g$ is continuous.

**Problem 5:** Let $f : X \to Y$ be any function and let $G_f \subseteq X \times Y$ be the graph of $f$. Prove that there exists a bijection between $X$ and $G_f$.

**Solution:** Define a function $\varphi : X \to G_f$ by $\varphi(x) = (x, f(x))$. Define a function $\pi : G_f \to X$ by $\pi(x, f(x)) = x$. For any $x \in X$ we have

$$\pi \circ \varphi(x) = \pi(x, f(x)) = x.$$  

For any $(x, f(x)) \in G_f$ we have

$$\varphi \circ \pi(x, f(x)) = \varphi(x) = (x, f(x)).$$

We conclude that $\pi$ and $\varphi$ are mutually inverse functions and, in particular, $\varphi$ is a bijection.

**Problem 6:** Let $f : X \to Y$ be a function. Prove that $f$ is surjective if and only if there exists a function $g : Y \to X$ such that $f \circ g = I_Y$. Is $g$ necessarily unique in this case?

**Solution:** Suppose $f$ is surjective. We define a function $g : Y \to X$ as follows. For any $y \in Y$, there is at least one pre-image of $y$ in $X$. Let $x_y$ be one choice of pre-image, so that $f(x_y) = y$. We set $g(y) = x_y$. For any $y \in Y$ we have

$$f \circ g(y) = f(x_y) = y = I_Y(y),$$

so that $f \circ g = I_Y$.

Suppose there exists a function $g : Y \to X$ such that $f \circ g = I_Y$. For any $y \in Y$, we have that

$$f(g(y)) = f \circ g(y) = I_Y(y) = y,$$

so that $f$ is surjective.

$g$ is not necessarily unique in this case. To see this, define $f : \{1, 2\} \to \{3\}$ by $f(1) = f(2) = 3$. We have two functions $g, g' : \{3\} \to \{1, 2\}$ defined by $g(3) = 1$ and $g'(3) = 2$. Moreover, we have

$$f \circ g = f \circ g' = I_{\{3\}}.$$

**Problem 7:** As of the beginning of this quarter, UCSD had 2016 math majors. All of these math majors must take an algebra class (Math 100 or Math 103), an analysis
class (Math 140 or Math 142) and will certainly take a combinatorics class (Math 154 or Math 184). No student may take both of 100/103, 140/142, or 154/184. 2

600 of the majors take Math 100. 800 of the majors take Math 140. 700 of the majors take Math 154. 300 majors take both Math 100 and Math 140. 300 majors take both Math 100 and Math 154. 400 majors take both Math 140 and Math 154. 100 majors take all of Math 100, 140, and 154. How many majors take none of Math 100, 140, and 154?

Solution: Given a course number \( x \), let \( A_x \) be the set of math majors which take course \( x \). The Principle of Inclusion-Exclusion gives

\[
|A_{100} \cup A_{140} \cup A_{154}| = |A_{100}| + |A_{140}| + |A_{154}| - |A_{100} \cap A_{140}| - |A_{100} \cap A_{154}| - |A_{140} \cap A_{154}| + |A_{100} \cap A_{140} \cap A_{154}|
\]

\[
= 600 + 800 + 700 - 300 - 300 - 400 + 100
\]

\[
= 1200.
\]

Since we are interested in the students which lie in the complement of \( A_{100} \cup A_{140} \cup A_{154} \), we get that \( 2016 - 1200 = 816 \) students take none of these classes.

Problem 8: Prove that a composition of surjections is a surjection and that a composition of injections is an injection.

Solution: Let \( f : X \to Y \) and \( g : Y \to Z \) be two functions.

Suppose \( f \) and \( g \) are surjections. Let \( z \in Z \). Since \( g \) is a surjection there exists \( y \in Y \) so that \( g(y) = z \). Since \( f \) is a surjection there exists \( x \in X \) so that \( f(x) = y \). Then

\[
g \circ f(x) = g(y) = z,
\]

so that \( g \circ f \) is a surjection.

Suppose \( f \) and \( g \) are injections. Let \( x_1, x_2 \in X \) with \( x_1 \neq x_2 \). Since \( f \) is an injection, we have \( f(x_1) \neq f(x_2) \). Since \( g \) is an injection, we have \( g(f(x_1)) \neq g(f(x_2)) \). This means \( g \circ f(x_1) \neq g \circ f(x_2) \), so that \( g \circ f \) is an injection.

\[2\]These aren’t quite the math major requirements.