Math 109: Spring 2016
Homework 4 Solutions
Due 5:00pm on Friday 5/6/2016

Problem 1: Prove that a finite non-empty set of real numbers contains a minimum element.

Solution: Let $X$ be a finite non-empty set of real numbers with $|X| = n$. We induct on $n$.

If $n = 1$, then $|X| = \{x\}$ for some $x \in \mathbb{R}$, so that $x$ is the minimum element of $X$.

If $n > 1$, write $X = \{x_1, \ldots, x_{n-1}, x_n\}$ for some $x_1, \ldots, x_{n-1}, x_n \in \mathbb{R}$. By induction, the set $\{x_1, \ldots, x_{n-1}\}$ has a minimum element $x_i$ for some $1 \leq i \leq n - 1$. If $x_i < x_n$, then $x_i$ is the minimum element of $X$. If $x_i > x_n$, then $x_n$ is the minimum element of $X$. In either case, we conclude that $X$ has a minimum element.

Problem 2: Let $A$ and $B$ be finite sets of a real numbers with $A \subseteq B$. Prove that $\min B \leq \min A \leq \max A \leq \max B$.

Solution: We have that $a \leq \max A$ for all $a \in A$. In particular, we have $\min A \leq \max A$. Since $\min A \in B$ and $\min B \leq b$ for all $b \in B$, we have $\min B \leq \min A$. Since $\max A \in B$ and $b \leq \max B$ for all $b \in B$, we have $\max A \leq \max B$. Putting these inequalities together gives $\min B \leq \min A \leq \max A \leq \max B$, as desired.

Problem 3: Let $n$ be a positive integer. Prove that the number of ordered pairs $(A, B)$ of subsets of $\mathbb{N}_n$ which satisfy $A \subseteq B$ is $3^n$.

Hint: Let $X$ denote the set of such ordered pairs. Try to define a bijection $\varphi : X \rightarrow \text{Fun}(\mathbb{N}_n, \mathbb{N}_3)$.

Solution: By a result proven in class, we have that $|\text{Fun}(\mathbb{N}_n, \mathbb{N}_3)| = 3^n$. Therefore, it suffices to construct a bijection $\varphi : X \rightarrow \text{Fun}(\mathbb{N}_n, \mathbb{N}_3)$, where $X = \{(A, B) \mid A \subseteq B \subseteq \mathbb{N}_n\}$.

For any $(A, B) \in X$, we define a function $\chi_{(A,B)} : \mathbb{N}_n \rightarrow \mathbb{N}_3$ as follows. For $1 \leq i \leq n$, set

$$\chi_{(A,B)}(i) = \begin{cases} 1 & i \in A \\ 2 & i \in B - A \\ 3 & i \in \mathbb{N}_n - B. \end{cases}$$

We define $\varphi : X \rightarrow \text{Fun}(\mathbb{N}_n, \mathbb{N}_3)$ by $\varphi(A, B) = \chi_{(A,B)}$. To prove that $\varphi$ is a bijection, we show that it has an inverse.

We define a function $\psi : \text{Fun}(\mathbb{N}_n, \mathbb{N}_3) \rightarrow X$.
as follows. Given a function $f : \mathbb{N}_n \to \mathbb{N}_3$, we let
\[
\psi(f) = (\{1 \leq i \leq n \mid f(i) = 1\}, \{1 \leq i \leq n \mid f(i) = 1 \text{ or } 2\}).
\]

We now check that $\varphi$ and $\psi$ are mutually inverse functions. Let $(A, B) \in X$. We have that
\[
\psi \circ \varphi(A, B) = \psi(\chi(A, B)) = (\{1 \leq i \leq n \mid \chi(A, B)(i) = 1\}, \{1 \leq i \leq n \mid \chi(A, B)(i) = 1 \text{ or } 2\}) = (A, B),
\]
where the last equality used the definition of the function $\chi(A, B)$. This means that $\psi \circ \varphi = I_X$.

On the other hand, let $f \in \text{Fun}(\mathbb{N}_n, \mathbb{N}_3)$. We have that
\[
\varphi \circ \psi(f) = \varphi(\{1 \leq i \leq n \mid f(i) = 1\}, \{1 \leq i \leq n \mid f(i) = 1 \text{ or } 2\}) = \chi(\{1 \leq i \leq n \mid f(i) = 1\}, \{1 \leq i \leq n \mid f(i) = 1 \text{ or } 2\}) = f,
\]
where the last equality used the definition of $\chi(A, B)$. This means that $\varphi \circ \psi = I_{\text{Fun}(\mathbb{N}_n, \mathbb{N}_3)}$, so that $\varphi$ is invertible.

**Problem 4:** Let $X$ and $Y$ be disjoint (not necessarily finite!) sets and let $k$ be a positive integer. Construct a bijection
\[
\varphi : \bigcup_{i=0}^{k} \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y) \to \mathcal{P}_k(X \cup Y).
\]
Explain what this has to do with the Vandermonde convolution:
\[
\sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} = \binom{n+m}{k}.
\]

**Solution:** We define the map
\[
\varphi : \bigcup_{i=0}^{k} \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y) \to \mathcal{P}_k(X \cup Y)
\]
by letting
\[
\varphi(A, B) = A \cup B.
\]
Since $X \cap Y = \emptyset$, for any finite sets $A \subseteq X$ and $B \subseteq Y$ we have $|A \cup B| = |A| + |B|$, so that $\varphi$ is well-defined.

To check that $\varphi$ is bijective, it is enough to construct its inverse. We define a function
\[
\psi : \mathcal{P}_k(X \cup Y) \to \bigcup_{i=0}^{k} \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y)
\]
as follows. Given $C \subseteq X \cup Y$ with $|C| = k$, we let
\[
\psi(C) = (C \cap X, C \cap Y).
\]
Since $X \cap Y = \emptyset$, we have $|C| = |C \cap X| + |C \cap Y|$ so that $\psi$ is well-defined.

We check that $\varphi$ and $\psi$ are mutually inverse functions. Indeed, suppose $(A, B) \in \bigcup_{i=0}^{k} P_i(X) \times P_{k-i}(Y)$. Then

$$\psi \circ \varphi(A, B) = \psi(A \cup B) = ((A \cup B) \cap X, (A \cup B) \cap Y) = (A, B).$$

On the other hand, if $C \subseteq X \cup Y$ and $|C| = k$ we have

$$\varphi \circ \psi(C) = \varphi(C \cap X, C \cap Y) = (C \cap X) \cup (C \cap Y) = C.$$

We conclude that $\varphi$ and $\psi$ are mutually inverse, and hence bijections.

Suppose that $X, Y$ are finite and $|X| = n, |Y| = m$. Taking cardinalities of the domain and codomain of $\varphi$ and using the fact that $\varphi$ is a bijection gives

$$\sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} = \binom{n+m}{k}.$$

**Problem 5:** Let $X$ be a finite set with $|X| = n$ and let $0 \leq k \leq n$. Construct a bijection

$$\varphi : P_k(X) \rightarrow P_{n-k}(X).$$

Explain what this has to do with the identity

$$\binom{n}{k} = \binom{n}{n-k}.$$

**Solution:** We define a map

$$\varphi : P_k(X) \rightarrow P_{n-k}(X)$$

by setting

$$\varphi(A) = A^c,$$

where $A^c = X - A$. If we let

$$\psi : P_{n-k}(X) \rightarrow P_k(X)$$

be defined by $\psi(B) = B^c = X - B$, we have that

$$\varphi \circ \psi(B) = \varphi(B^c) = (B^c)^c = B$$

and

$$\psi \circ \varphi(A) = \psi(A^c) = (A^c)^c = A$$

always. It follows that $\psi$ and $\varphi$ are mutually inverse functions, and so $\varphi$ is a bijection.

Taking the cardinality of the domain and codomain of $\varphi$ and using the fact that $\varphi$ is a bijection gives

$$\binom{n}{k} = \binom{n}{n-k}.$$

**Problem 6:** Let $X$ be a finite nonempty set with $|X| = n$ and let $1 \leq k \leq n$. Let $x_0 \in X$. Construct a bijection

$$\varphi : P_k(X) \rightarrow P_{k-1}(X - \{x_0\}) \cup P_k(X - \{x_0\}).$$
Explain what this has to do with the Pascal recursion:
\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.
\]

Solution: We define \(\varphi(A) = A - \{x_0\}\) for any \(A \in \mathcal{P}_k(X)\). To prove that \(\varphi\) is a bijection, we construct its inverse. Define
\[
\psi : \mathcal{P}_k(X - \{x_0\}) \cup \mathcal{P}_{k-1}(X - \{x_0\}) \longrightarrow \mathcal{P}_k(X)
\]
by
\[
\psi(B) = \begin{cases} 
B & |B| = k \\
B \cup \{x_0\} & |B| = k - 1.
\end{cases}
\]

We claim that \(\varphi\) and \(\psi\) are inverse functions. To see this, consider \(A \in \mathcal{P}_k(X)\). We have
\[
\psi \circ \varphi(A) = \begin{cases} 
\psi(A) & x_0 \notin A \\
\psi(A - \{x_0\}) & x_0 \in A
\end{cases} = \begin{cases} 
A & x_0 \notin A \\
(A - \{x_0\}) \cup \{x_0\} & x_0 \in A
\end{cases} = A.
\]

Now consider \(B \in \mathcal{P}_k(X - \{x_0\}) \cup \mathcal{P}_{k-1}(X - \{x_0\})\). We have
\[
\varphi \circ \psi(B) = \begin{cases} 
\varphi(B) & |B| = k \\
\varphi(B \cup \{x_0\}) & |B| = k - 1
\end{cases} = \begin{cases} 
B & |B| = k \\
(B \cup \{x_0\}) - \{x_0\} & |B| = k - 1
\end{cases} = B.
\]

Thus \(\varphi\) and \(\psi\) are inverse functions, so that \(\varphi\) is a bijection.

Since \(\varphi\) is a bijection, the cardinalities of its domain and codomain are equal. Since \(|X| = n\), we get the Pascal recursion
\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.
\]

Problem 7: We are given 17 points inside a regular triangle of side length 1. Prove that there are two points among them whose distance is not more than 1/4. (Hint: Subdivide the triangle in a clever way and apply the Pigeonhole Principle.)

Solution: Let \(T\) be the regular triangle in question. We may subdivide \(T\) into four regular triangles \(T'_1, \ldots, T'_4\) of side length 1/2 by inscribing a regular triangle whose vertices are the midpoints of the sides of \(T\). Repeating this procedure with each of the \(T'_i\), we get 16 regular triangles \(T''_1, \ldots, T''_{16}\) of side length 1/4 which subdivide \(T\). By the Pigeonhole Principle, there exists a value \(1 \leq i \leq 16\) such that \(T''_i\) contains (at least) two of the 17 points in question. Since \(T''_i\) is a regular triangle of side length 1/4, these two points cannot have distance > 1/4.