**Problem 1:** A polynomial with coefficients in $\mathbb{Q}$ is a finite expression of the form
\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]
where $a_n \neq 0$. The number $n$ is called the degree of $f(x)$.

(1) Fix a non-negative integer $n$. Prove that the set $P_n$ of polynomials of degree $n$ with coefficients in $\mathbb{Q}$ is countable.

(2) Use Part (1) to prove that the set $P$ of all polynomials with coefficients in $\mathbb{Q}$ is countable.

**Solution:**

(1) We proved in class that $\mathbb{Q}$ is countable. Since subsets of countable sets are countable, we know that $\mathbb{Q} - \{0\}$ is also countable. Since finite products of countable sets are countable, we know that $(\mathbb{Q} - \{0\}) \times \mathbb{Q}^n$ is countable. It is evident that the map
\[ \varphi : P_n \to (\mathbb{Q} - \{0\}) \times \mathbb{Q}^n \]
given by
\[ \varphi(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) = (a_n, a_{n-1}, \ldots, a_1, a_0) \]
is a bijection. We conclude that $P_n$ is countable.

(2) We have that
\[ P = \bigcup_{n \geq 0} P_n. \]
By Part (1), $P_n$ is countable for each $n \geq 0$. Since countable unions of countable sets are countable, we conclude that $P$ is countable.

**Problem 2:** A complex number $x_0 \in \mathbb{C}$ is called algebraic if there is a non-zero polynomial $f(x)$ with coefficients in $\mathbb{Q}$ such that $f(x_0) = 0$.

(1) Prove that every rational number is algebraic.

(2) Prove that $\sqrt{2}$ is algebraic. (It can be shown that $\sqrt{2}$ is not rational.)

**Solution:**

(1) Let $a \in \mathbb{Q}$ be a rational number. Then $f(x) = x - a$ is a non-zero polynomial with coefficients in $\mathbb{Q}$ and we have $f(a) = a - a = 0$. We conclude that $a$ is algebraic.

(2) Let $g(x) = x^2 - 2$. Then $g(x)$ is a non-zero polynomial with coefficients in $\mathbb{Q}$ and we have $g(\sqrt{2}) = (\sqrt{2})^2 - 2 = 2 - 2 = 0$. We conclude that $\sqrt{2}$ is algebraic.

**Problem 3:** Prove that the set $A$ of algebraic numbers is countable.
(Hint: You may use without proof the fact that, if $f(x)$ is a non-zero polynomial with coefficients in $\mathbb{Q}$ of degree $n$, the equation $f(x) = 0$ has at most $n$ solutions. Now use the fact that the set $P_n$ of Problem 1 is countable.)

**Solution:** By Problem 1, the set $P$ of all non-zero polynomials with coefficients in $\mathbb{Q}$ is countable. Write

$$P = \{f_1(x), f_2(x), \ldots\}.$$  

For all $n \geq 1$, let $Z_n = \{a \in \mathbb{C} \mid f(a) = 0\}$. By the fact referenced in the problem, the set $Z_n$ is finite for all $n \geq 1$. Moreover, we have that

$$A = \bigcup_{n \geq 1} Z_n.$$  

Since a countable union of countable sets is countable, we conclude that $A$ is countable.

**Problem 4:** A complex number $x \in \mathbb{C}$ is called *transcendental* if $x$ is not algebraic. Prove that the set $T$ of transcendental numbers is uncountable.

(Remark: It is very hard to prove that a given complex number $x$ is transcendental. Essentially the only known examples are $e$ and $\pi$. We do not even know if $\pi^\pi$ is transcendental. However, this problem shows that there are “more” transcendental numbers than algebraic numbers.)

**Solution:** By definition we have that $\mathbb{C} = A \cup T$. We proved in class that $\mathbb{R}$ is uncountable. Since $\mathbb{R} \subset \mathbb{C}$ we conclude that $\mathbb{C}$ is uncountable. By Problem 3 we have that $A$ is countable. Since a countable union of countable sets is countable, this forces $T$ to be uncountable.

**Problem 5:** An *infinite binary sequence* is a expression of the form

$$(a_1, a_2, a_3, \ldots)$$

where each $a_n$ is in $\{0, 1\}$. Prove that the set $B$ of all infinite binary sequences is uncountable.

(Hint: Modify Cantor’s diagonalization argument.)

**Solution:** Suppose $B$ were countable. Write $B = \{b^{(1)}, b^{(2)}, \ldots\}$. For each $n \geq 1$, list the terms of $b^{(n)}$ as

$$b^{(n)} = (b_1^{(n)}, b_2^{(n)}, \ldots).$$

Define a new binary sequence $b \in B$ by $b = (b_1, b_2, \ldots)$, where

$$b_n = \begin{cases} 1 & b_n^{(n)} = 0 \\ 0 & b_n^{(n)} = 1. \end{cases}$$

For every $n \geq 1$, we have that $b$ disagrees with $b^{(n)}$ in its $n^{th}$ term, so that $b \neq b^{(n)}$. Hence, we have $b \in B - \{b^{(1)}, b^{(2)}, \ldots\}$, which is a contradiction.

**Problem 6:** Let $X$ and $Y$ be sets and let $\text{Fun}(X,Y)$ be the set of all functions from $X$ to $Y$. Prove or disprove the following two statements.

1. If $X$ is countable and $Y$ is finite, then $\text{Fun}(X,Y)$ is countable.
2. If $X$ is finite and $Y$ is countable, then $\text{Fun}(X,Y)$ is countable.
Solution:

(1) This is false. Take $X = \mathbb{Z}^+$ and $Y = \{0, 1\}$. Then elements in $\text{Fun}(X, Y)$ are exactly the same thing as infinite binary sequences. By Problem 5, we conclude that $\text{Fun}(X, Y)$ is uncountable.

(2) This is true. Write $X = \{x_1, \ldots, x_n\}$. Define a map

$$\varphi : \text{Fun}(X, Y) \to Y^n$$

by

$$\varphi(f) = (f(x_1), \ldots, f(x_n)).$$

We claim that $\varphi$ is a bijection. Indeed, consider the function

$$\psi : Y^n \to \text{Fun}(X, Y)$$

given by $\psi(y_1, \ldots, y_n) : x_i \mapsto y_i$ for $1 \leq i \leq n$. It is easily seen that $\psi$ is the inverse map to $\varphi$.

Since finite products of countable sets are countable, we have that $Y^n$ is countable. Since $\varphi$ is a bijection, we also have that $\text{Fun}(X, Y)$ is countable.