Math 109: Spring 2016
Midterm 2 Solutions

Instructions: Please write your name and section number on your blue book. Make it clear in your blue book what problem you are working on. Write legibly and explain your reasoning. This exam is graded out of 100 points. Following these instructions is worth 5 points.

Problem 1: [15] Let \( f : X \to Y \) be a function between sets. Suppose there exists a function \( g : Y \to X \) such that \( g \circ f = I_X \). Prove that \( f \) is injective.

Solution: Let \( x_1, x_2 \in X \) and suppose \( f(x_1) = f(x_2) \). We have
\[
I_X(x_1) = g(f(x_1)) = g(f(x_2)) = I_X(x_2) = x_2.
\]
We conclude that \( f \) is injective.

Problem 2: [15] Let \( X \) be a finite set and \( n \in \mathbb{Z}_{\geq 0} \). (a) Carefully state what it means to say “\( |X| = n \)”. (Hint: Your answer should involve a function.) (b) If \( X \) and \( Y \) are finite sets, \( |X| = n \), and \( \varphi : X \to Y \) is a bijection, prove that \( |Y| = n \).

Solution: (a) “\( |X| = n \)” means that there exists a bijection \( f : \mathbb{N}_n \to X \).
(b) Since \( |X| = n \), there exists a bijection \( f : \mathbb{N}_n \to X \). The composite map \( \varphi \circ f : \mathbb{N}_n \to Y \) is a composition of two bijections, and hence itself a bijection. It follows that \( |Y| = n \).

Problem 3: [15] Prove or disprove: For any finite sets \( X \) and \( Y \) we have \( |\mathcal{P}(X \times Y)| = |\mathcal{P}(X) \times \mathcal{P}(Y)| \).

Solution: This is false. Indeed, suppose \( X = \{1\} \) and \( Y = \{2, 3\} \). We have that
\[
\mathcal{P}(X \times Y) = \{\emptyset, \{(1, 2)\}, \{(1, 3)\}, \{(1, 2), (1, 3)\}\}.
\]
On the other hand, we have
\[
\mathcal{P}(X) \times \mathcal{P}(Y) = \{\emptyset, \emptyset, \emptyset, \emptyset\}, \{\emptyset, \{2\}, \{3\}, \emptyset\}, \{(1, Y), (X, \emptyset), (X, \{2\}), (X, \{3\})\}, (X, Y)\}.
\]
We conclude that \( |\mathcal{P}(X \times Y)| = 4 \) but \( |\mathcal{P}(X) \times \mathcal{P}(Y)| = 8 \).

Problem 4: [15] Observe that we can write the set \( \mathbb{Q} \) of rational numbers as
\[
\mathbb{Q} = \bigcup_{n \geq 1} \left\{-n, -n + \frac{1}{n}, -n + \frac{2}{n}, \ldots, n - \frac{1}{n}, n\right\}.
\]
Use this observation to prove that \( \mathbb{Q} \) is countable.

Solution: For any fixed integer \( n \geq 1 \), the set \( A_n = \{-n, -n + \frac{1}{n}, \ldots, n - \frac{1}{n}, n\} \) is finite. It follows that \( \mathbb{Q} = \bigcup_{n \geq 1} A_n \) is a countable union of finite sets. We proved in class that a countable union of finite sets (and indeed a countable union of countable sets) is countable. We conclude that \( \mathbb{Q} \) is countable.
**Problem 5:** [15] Let $S$ be a sphere and let $P_1, P_2, P_3, P_4, P_5$ be five points on $S$. Prove that there is a closed hemisphere $H$ on $S$ containing at least four of these points.

**Solution:** Draw a great circle $C$ through $P_4$ and $P_5$. The great circle $C$ divides $S$ into two closed hemispheres

$$S = H^+ \cup H^-.$$

By the Pigeonhole Principle, two of the remaining three points $P_1, P_2, P_3$ must lie in one of these closed hemispheres. Let $H$ be a hemisphere containing at least two of these three points. Since $H$ also contains $C$ (and hence $P_4$ and $P_5$), $H$ contains at least four of the points $P_1, \ldots, P_5$.

**Problem 6:** [20] Is $\text{Fun}(\mathbb{Z}^+, \{0, 1\})$ a countable set? Justify your answer.

**Solution:** No. Suppose $\text{Fun}(\mathbb{Z}^+, \{0, 1\})$ were countable. Then we could write

$$\text{Fun}(\mathbb{Z}^+, \{0, 1\}) = \{f_1, f_2, f_3, \ldots\}$$

for some collection of functions $\{f_n : \mathbb{Z}^+ \to \{0, 1\} \mid n = 1, 2, 3, \ldots\}$. We define a new function $g : \mathbb{Z}^+ \to \{0, 1\}$ by the rule

$$g(n) = \begin{cases} 1 & f_n(n) = 0 \\ 0 & f_n(n) = 1. \end{cases}$$

For any $n \in \mathbb{Z}^+$, we have that $g \neq f_n$ because $g(n) \neq f_n(n)$. It follows that $g \notin \{f_1, f_2, f_3, \ldots\}$, which contradicts the assumption $\text{Fun}(\mathbb{Z}^+, \{0, 1\}) = \{f_1, f_2, f_3, \ldots\}$.