Instructions: Please write your name and section number on your blue book. Make it clear in your blue book what problem you are working on. Write legibly and explain your reasoning. This exam is graded out of 100 points. Following these instructions is worth 5 points.

Problem 1: [10] Let P and Q be statements. Write down a truth table for the statement ‘[(P ⇒ Q) and Q] ⇒ P’.

Solution:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P ⇒ Q</th>
<th>(P ⇒ Q) and Q</th>
<th>[(P ⇒ Q) and Q] ⇒ P</th>
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Problem 2: [15] Prove or disprove: For any sets X and Y, we have that

\[ P(X \cap Y) = P(X) \cap P(Y). \]

Here \( P(A) \) denotes the power set of a set \( A \).

Solution: This is true. Let \( A \) be any set. We have that \( A \in P(X \cap Y) \) if and only if \( A \subseteq X \cap Y \). But \( A \subseteq X \cap Y \) if and only if \( A \subseteq X \) and \( A \subseteq Y \). This latter condition is equivalent to \( A \in P(X) \cap P(Y) \).

Problem 3: [10] Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be a function from the real numbers to itself and let \( x_0 \in \mathbb{R} \). We say that \( f \) is continuous at \( x_0 \) if for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for every \( x \in \mathbb{R} \) with \( |x - x_0| < \delta \), we have \( |f(x) - f(x_0)| < \varepsilon \). State what it means for \( f \) to not be continuous at \( x_0 \).

Solution: The function \( f \) is not continuous at \( x_0 \) if there exists \( \varepsilon > 0 \) such that for any \( \delta > 0 \) there exists \( x \in \mathbb{R} \) with \( |x - x_0| < \delta \) and \( |f(x) - f(x_0)| \geq \varepsilon \).

Problem 4: [20] Prove that there do not exist integers \( m \) and \( n \) such that

\[ 27m + 18n = 3. \]

Solution: Working towards a contradiction, suppose that there are integers \( m \) and \( n \) such that \( 27m + 18n = 3 \). Then \( 3 = 27m + 18n = 9(3m + 2n) \), which implies that \( 9 \mid 3 \). But we know that \( 9 \mid 3 \) is false. This contradiction shows that there do not exist integers \( m \) and \( n \) such that \( 27m + 18n = 3 \).
Problem 5: [20] Let $x$ and $y$ be distinct positive integers. Prove that we have
\[
\frac{x}{y} + \frac{y}{x} > 2.
\]

**Solution:** Let $x$ and $y$ be distinct positive integers. Since $x \neq y$ we have $x - y \neq 0$. This means that $0 < (x - y)^2 = x^2 - 2xy + y^2$. Rearranging gives $2xy < x^2 + y^2$. Dividing both sides by $xy$ (and applying the fact that $xy$ is positive) we get $2 < \frac{x}{y} + \frac{y}{x}$, as desired.

Problem 6: [20] Prove that for all positive integers $n$ we have
\[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.
\]

**Solution:** We induct on $n$. If $n = 1$, we have that $\sum_{i=1}^{n} i^2 = 1^2 = 1$ and $\frac{n(n+1)(2n+1)}{6} = \frac{1(2)(3)}{6} = 1$, so the desired equality is true in this case.

Fix an integers $n \geq 1$ and inductively assume that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. We have
\[
\sum_{i=1}^{n+1} i^2 = (n+1)^2 + \sum_{i=1}^{n} i^2
= (n+1)^2 + \frac{n(n+1)(2n+1)}{6}
= \frac{6n^2 + 12n + 6}{6} + \frac{2n^3 + 3n^2 + n}{6}
= \frac{2n^3 + 9n^2 + 13n + 6}{6}
= \frac{(n+1)(n+2)(2n+3)}{6}
= \frac{(n+1)((n+1) + 1)(2(n+1) + 1)}{6},
\]
where the second equality used the inductive hypothesis.

By induction, we have that the desired equality holds for all integers $n \geq 1$. 