Math 109: Winter 2014
Final Exam

Instructions: Please write your name on your blue book. Make it clear in your blue book what problem you are working on. Write legibly and explain your reasoning. This exam is graded out of 100 points. Following these instructions is worth 5 points.

Problem 1: [5 points] Construct a truth table for the statement “[not(P or Q)] ⇒ R”.

Solution:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>P or Q</th>
<th>not(P or Q)</th>
<th>[not (P or Q)] ⇒ R</th>
</tr>
</thead>
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</table>

Problem 2: [10 points] Prove that for all \( n \in \mathbb{Z}^+ \), we have \( 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \).

Solution: We induct on \( n \). If \( n = 1 \), we have \( 1^2 = 1 = \frac{1(2)(3)}{6} \), so the claim holds in this case.

Let \( n > 1 \) and inductively assume that

\[
1^2 + 2^2 + \cdots + (n-1)^2 = \frac{(n-1)n(2n-1)}{6}.
\]

We have that

\[
1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = (1^2 + 2^2 + \cdots + (n-1)^2) + n^2
\]

\[
= \frac{(n-1)n(2n-1)}{6} + n^2
\]

\[
= \frac{2n^3 - 3n^2 + n}{6} + \frac{6n^2}{6}
\]

\[
= \frac{2n^3 + 3n^2 + n}{6}
\]

\[
= \frac{n(n+1)(2n+1)}{6}.
\]

By induction, we conclude that \( 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \) for all \( n \in \mathbb{Z}^+ \).

Problem 3: [10 points] Prove that two different people on Facebook have the same number of friends.
Solution: Suppose there are \( n \) people on Facebook. Define a function

\[ f : \{ \text{all people on Facebook} \} \to \{ 0, 1, \ldots, n - 1 \} \]

by letting \( f(p) \) be the number of friends a person \( p \) has on Facebook. By the Pigeonhole Principle, if \( f \) is not surjective, then \( f \) must not be injective, i.e. two people must have the same number of friends on Facebook. We may therefore assume that \( f \) is surjective.

This means there is a person \( p \) with \( f(p) = 0 \) and another person \( p' \) with \( f(p') = n - 1 \). But then \( p \) is friendless whereas \( p' \) is friends with everyone (and in particular \( p \)). This is a contradiction.

Problem 4: [10 points] A function \( f : \mathbb{R} \to \mathbb{R} \) is said to be uniformly continuous if for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for all \( x, y \in \mathbb{R} \) with \( |x - y| < \delta \) we have that \( |f(x) - f(y)| < \epsilon \). Carefully state what it means for \( f \) to not be uniformly continuous.

Solution: \( f \) is not uniformly continuous if there exists \( \epsilon > 0 \) such that for all \( \delta > 0 \) there exist \( x, y \in \mathbb{R} \) with \( |x - y| < \delta \) and \( |f(x) - f(y)| \geq \epsilon \).

Problem 5: [5 + 10 points] (a) Define what it means for a set to be “denumerable”. (Your definition should involve a function.) (b) Is \([0,1] = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \}\) denumerable? Prove your claim.

Solution: (a) \( X \) is denumerable if there is a bijection \( f : \mathbb{Z}^+ \to X \).

(b) No. If \([0,1] \) were denumerable, write \([0,1] = \{ x_1, x_2, \ldots \}\). For each \( n \in \mathbb{Z}^+ \), let \( x_n = 0.b_1^{(n)}b_2^{(n)}b_3^{(n)} \ldots \) be the decimal expansion of \( x_n \). Define a new number \( y \in [0,1] \) by \( y = 0.c_1c_2c_3 \ldots \), where

\[
c_n = \begin{cases} 7 & b_n^{(n)} \neq 7 \\ 4 & b_n^{(n)} = 7. \end{cases}
\]

For any \( n \in \mathbb{Z}^+ \), the decimal expansion of \( y \) disagrees with the decimal expansion of \( x_n \) in the \( n^{th} \) position, so that \( y \neq x_n \). Therefore \( y \notin \{ x_1, x_2, \ldots \} \), which contradicts \([0,1] = \{ x_1, x_2, \ldots \}\).

Problem 6: [10 points] Accurately state the Division Theorem.

Solution: If \( a \in \mathbb{Z} \) and \( b \in \mathbb{Z}^+ \), there exist unique \( q, r \in \mathbb{Z} \) with \( 0 \leq r < b - 1 \) such that

\[ a = qb + r. \]

Problem 7: [10 points] Let \( a \) be an integer. Prove that \( 5 | a \) if and only if \( 5 | a^2 \).

Solution: Suppose \( 5 | a \). Then there exists \( k \in \mathbb{Z} \) with \( a = 5k \), so that \( a^2 = 5(5k^2) \). We conclude that \( 5 | a^2 \).
Suppose $5 \nmid a$. By a result proven in class (which relied on the Division Theorem), we have $a = 5k + 1, 5k + 2, 5k + 3,$ or $5k + 4$ for some $k \in \mathbb{Z}$. Hence,

$$a^2 = \begin{cases} 5(5k^2 + 2k) + 1 & a = 5k + 1 \\ 5(5k^2 + 4k) + 4 & a = 5k + 2 \\ 5(5k^2 + 6k + 1) + 4 & a = 5k + 3 \\ 5(5k^2 + 8k + 3) + 1 & a = 5k + 4. \end{cases}$$

In any case, we have that $a^2 = 5k' + 1$ or $5k' + 4$ for some $k' \in \mathbb{Z}$. Again by a result proven in class (which relied on the Division Theorem), we have $5 \nmid a^2$.

**Problem 8:** [5 + 5 + 5 points] Give examples of sets $X$ with binary relations $\sim$ satisfying the specified conditions. You do not need to prove that your examples work.

1. The binary relation $\sim$ on $X$ is reflexive and symmetric, but not transitive.
2. The binary relation $\sim$ on $X$ is reflexive and transitive, but not symmetric.
3. The binary relation $\sim$ on $X$ is symmetric and transitive, but not reflexive.

**Solution:**

1. Let $X = \{1, 0, -1\}$ and define $\sim$ on $X$ by $a \sim b$ if and only if $ab \geq 0$.
2. Let $X = \mathbb{Z}^+$ and define $\sim$ on $X$ by $a \sim b$ if and only if $a \mid b$.
3. Let $X = \{0\}$ and define $\sim$ on $X$ by $0 \not\sim 0$.

**Problem 9:** [10 points] Prove or disprove: For any set $X$, there exists an injection $f : X \to \mathcal{P}(X)$.

**Solution:** This is true. Define $f : X \to \mathcal{P}(X)$ by

$$f(x) = \{x\}.$$ 

If $x \neq y$, then $\{x\} \neq \{y\}$, so that $f$ is an injection.