Math 154: Winter 2017
Homework 3
Due 5:00pm on Wednesday 2/1/2017

Problem 1: Let G be a simple graph. The *complement* $\overline{G}$ of G has the same vertex set as G, and edges given by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. A graph is called *self-complementary* if it is isomorphic to its own complement.

- Prove that, if G is self-complementary, then G has $4k$ or $4k+1$ vertices for some integer k.
- Find all self-complementary graphs with 4 and 5 vertices.
- Find a self-complementary graph with 8 vertices.

Problem 2: Let G be a simple graph. The *line graph* $L(G)$ has vertex set $V(L(G)) = E(G)$ given by the *edge set* of G. Two elements $e, e' \in V(L(G))$ are adjacent if and only if e and e' share a vertex in G.

- Show that K_3 and $K_{1,3}$ have the same line graph.
- Suppose G is regular of degree k. Prove that $L(G)$ is regular of degree $2k-2$.
- Find an expression for the number of edges of $L(G)$ in terms of the degree sequence of G.

Problem 3: Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$ and $n \times n$ adjacency matrix A. Prove that the (i,j)-entry of A^k is the number of length k walks from v_i to v_j in G, for all $k \geq 0$.

Problem 4: Let G be a graph and suppose C_1 and C_2 are distinct cycles in G which both contain some edge e. Prove that G has a cycle which does not contain e.

Problem 5: Let G be a graph and suppose C_1^* and C_2^* are distinct cutsets in G which both contain some edge e. Prove that G has a cutset which does not contain e.

Problem 6: Let G be a connected graph, let C be a cycle in G, and let C^* be a cutset in G. Prove that C and C^* have an even number of edges in common.

Problem 7: Let G be a connected graph. For any vertices $u, v \in V(G)$, the *distance* $d(u,v)$ is defined by

$$d(u,v) = \text{length of the shortest walk from } u \text{ to } v.$$

Prove that

- $d(u,v) \geq 0$ for all $u, v \in V(G)$, with equality if and only if $u = v$.
- $d(u,v) = d(v,u)$ for all $u, v \in V(G)$,
- $d(u,v) \leq d(u,w) + d(w,v)$ for all $u, v, w \in V(G)$.

You have just shown that the function $d : V(G) \times V(G) \to \mathbb{R}_{\geq 0}$ is a *metric* on $V(G)$.