Math 154: Winter 2017
Homework 4
Due 5:00pm on Wednesday 2/15/2017

Problem 1: Let G be a simple Eulerian graph. Prove that the line graph $L(G)$ of G is Eulerian. If G is a simple graph such that $L(G)$ is Eulerian, is G necessarily Eulerian?

Problem 2: Let us call a graph G *semi-Eulerian* if there is a trail $v_0 \to \cdots \to v_k$ in G which uses every edge in G, but has $v_0 \neq v_k$. (Such a trail is a *semi-Eulerian trail.*) Prove that a connected graph G is semi-Eulerian if and only if G contains precisely two vertices of odd degree.

Problem 3: Let G be a Hamiltonian graph and let S be a set of k vertices in G. Prove that $G - S$ has at most k components.

Problem 4: Let G be a connected graph. What can you say about

- an edge e in G which appears in every spanning tree?
- an edge e in G which appears in no spanning tree?

Problem 5: How many spanning trees does the complete bipartite graph $K_{2,s}$ have?

Problem 6: Let $\tau(G)$ be the number of spanning trees in a connected graph G. Prove that, for any edge e in G, we have $\tau(G) = \tau(G - e) + \tau(G \setminus e)$.

Problem 7: Find the number of labeled trees on $1, 2, \ldots, n$ for which 1 is an end-vertex.

Problem 8: Let G be a connected simple graph with vertex set $\{v_1, \ldots, v_n\}$. Define an $n \times n$ matrix $M = (m_{i,j})$ by $m_{i,i} = \deg(v_i), m_{i,i} = -1$ if v_i and v_j are adjacent, and $m_{i,j} = 0$ otherwise. (The matrix M is called the *Laplacian* of the graph G.) The *Matrix-Tree Theorem* states that the number of spanning trees of G is equal to any cofactor of M. Use the Matrix-Tree Theorem to prove that the number of spanning trees of K_n is n^{n-2}.