Problem 1: Let $G$ be a polyhedron (or polyhedral graph), each of whose faces is bounded by a pentagon or a hexagon. Use Euler’s Formula to prove that $G$ must have at least 12 pentagonal faces. Prove that if, in addition, $G$ has exactly three faces meeting at each vertex, then $G$ has exactly 12 pentagonal faces.

Solution: Let $n$ be the number of vertices of $G$, $m$ be the number of edges of $G$, $f_5$ be the number of pentagonal faces of $G$, and $f_6$ be the number of hexagonal faces of $G$. Since every edge is contained in precisely two faces, we have

$$5f_5 + 6f_6 = 2m.$$ 

Moreover, since $G$ is polyhedral, we know that each vertex is incident to at least 3 faces, so that

$$3n \leq 5f_5 + 6f_6.$$ 

Finally, we have Euler’s Formula:

$$n - m + f_5 + f_6 = 2.$$ 

We want to show that $f_5 \geq 12$. To see this, multiply both sides Euler’s Formula by 6 and use $6m = 15f_5 + 18f_6$ and $6n \leq 10f_5 + 12f_6$ to get

$$(10f_5 + 12f_6) - (15f_5 + 18f_6) + (6f_5 + 6f_6) \geq 12,$$

or $f_5 \geq 12$.

If exactly three faces meet at each vertex, we have $3n = 5f_5 + 6f_6$, so that the above chain of reasoning implies $f_5 = 12$.

Problem 2: Let $G$ be a simple plane graph with fewer than 12 faces, in which each vertex has degree at least 3. Use Euler’s Formula to prove that $G$ has a face bounded by at most 4 edges. Give an example to show that this can fail if $G$ has 12 faces.

Solution: As usual, let $(n, m, f)$ denote the number of (vertices, edges, faces) of $G$. Since each vertex has degree at least 3, we have

$$3n \leq 2m.$$ 

If every face of $G$ were bounded by $\geq 5$ edges we would have

$$5f \leq 2m.$$ 

If we multiply both sides of Euler’s Formula $n - m + f = 2$ by 6, we get

$$6n - 6m + 6f = 12.$$ 

Applying the above inequalities gives

$$4m - 6m + 6f \geq 12,$$

so that

$$-5f + 6f \geq 12.$$ 

Therefore, $f \leq 4$. If $G$ has 12 faces, then $f = 12$, $m = 22$, and $n = 14$, so that $f \neq 4$.
and $f \geq 12$, which is a contradiction.

To see that this can fail if $G$ has 12 faces, let $G$ be the plane graph of the dodecahedron. Every face of $G$ has precisely 5 edges, every vertex of $G$ has degree 3, and $G$ has 12 faces.

**Problem 3:** Let $G$ be a Hamiltonian graph and let $S$ be a set of $k$ vertices in $G$. Prove that $G - S$ has at most $k$ components.

**Solution:** (Repeated from previous homework.)

**Problem 4:** The $k$-cube $Q_k$ has $2^k$ vertices given by

$$V(Q_k) = \{(x_1, \ldots, x_k) : x_i = 0 \text{ or } 1 \}$$

and an edge connecting two vertices $(x_1, \ldots, x_k)$ and $(x'_1, \ldots, x'_k)$ if and only if these vertices differ in exactly one coordinate. For which values of $k$ is $Q_k$ planar?

**Solution:** We claim that $Q_k$ is planar if and only if $k \leq 3$. It is clear that $Q_1$, $Q_2$, and $Q_3$ are planar. Since $Q_k$ contains a subgraph isomorphic to $Q_{k'}$ for all $k > k'$, it is enough to show that $Q_4$ is not planar.

The graph $Q_4$ has $2^4 = 16$ vertices. Since each vertex in $Q_4$ has degree 4, we see that $Q_4$ has $\frac{1}{2}(4 \times 16) = 32$ edges. Moreover, the graph $Q_4$ contains no triangles (indeed, the graph $Q_4$ is bipartite – a bipartition $(A, B)$ of the vertex set is obtained by letting $A$ be the set of vertices with odd coordinate sum and $B$ be the set of vertices with even coordinate sum). If $Q_4$ were planar we would have

$$32 \leq 2(16) - 4,$$

which is false.

**Problem 5:** The complete tripartite graph $K_{r,s,t}$ consists of three sets of vertices

$$\{a_1, \ldots, a_r\} \cup \{b_1, \ldots, b_s\} \cup \{c_1, \ldots, c_t\}$$

with edges connecting $a_ib_j, a_ic_j$, and $b_ic_j$ for all $i$ and $j$. For which values of $r, s, t$ is $K_{r,s,t}$ planar?

**Solution:** Without loss of generality we may assume $r \geq s \geq t$. If $r \geq 3$ and $(s + t) \geq 3$, the graph $K_{r,s,t}$ contains a copy of $K_{3,3,3}$, so is not planar. We claim that $K_{r,s,t}$ is planar in all other cases.

If $r \leq 2$ and $(s + t) \leq 4$ it suffices to exhibit a planar drawing of $K_{2,2,2}$. Such a drawing may be found here:


If $r \geq 3$ and $(s + t) \leq 2$ it suffices to exhibit a planar drawing of $K_{r,1,1}$. We obtain this drawing by placing $r$ vertices $a_1, \ldots, a_r$ on the $x$-axis, placing $b_1$ and the point $(0,1)$, and placing $c_1$ at the point $(0,-1)$. Draw straight line segments between the vertices $a_ib_1$ and $a_ic_1$ for $1 \leq i \leq r$. Then draw a curved line segment between $b_1$ and $c_1$ which does not intersect any of the other line segments.
Problem 6: By placing vertices at 

(1, 1^2, 1^3), (2, 2^2, 2^3), (3, 3^2, 3^3), \ldots \text{ prove that any simple graph can be drawn without crossings in three-dimensional space with every edge represented by a straight line.}

Solution: It is enough to show that the line segment \( \ell \) connecting \((a, a^2, a^3), (b, b^2, b^3)\) and \( \ell' \) connecting \((c, c^2, c^3), (d, d^2, d^3)\) (for \(a \neq b\) and \(c \neq d\)) do not intersect unless two of the numbers \(a, b, c, d\) coincide. To do this, it suffices to show that the four points \((a, a^2, a^3), \ldots, (d, d^2, d^3)\) are not coplanar for \(a, b, c, d\) distinct. This is equivalent to the matrix

\[
A = \begin{pmatrix}
a^3 & b^3 & c^3 & d^3 \\
a^2 & b^2 & c^2 & d^2 \\
a & b & c & d \\
1 & 1 & 1 & 1
\end{pmatrix}
\]

being invertible. A direct calculation (or an invocation of the Vandermonde determinant formula) shows

\[
\det(A) = (a - b)(a - c)(a - d)(b - c)(b - d)(c - d),
\]

which is nonzero since \(a, b, c, d\) are distinct.

Problem 7: Find a drawing of \(K_{3,3}\) on the torus in which edges do not cross.

Solution: Check it out:

http://www.cut-the-knot.org/blue/JCT/K33onTorus.shtml

They even draw \(K_{3,3}\) on the Möbius band!

Problem 8: Use duality to show that there is no plane graph with five faces, each of which share an edge in common.

Solution: Suppose \(G\) were a plane drawing of such a graph; let \(G^*\) be its geometric dual. If \(F_1, \ldots, F_5\) were five faces of \(G\) which all shared an edge in common, then \(F_1, \ldots, F_5\) would become five vertices of \(G^*\), each sharing an edge. That is, the graph \(G^*\) contains a subgraph isomorphic to \(K_5\). However, the graph \(K_5\) is not planar.

Problem 9: What is the chromatic number of the \(k\)-cube \(Q_k\)?

Solution: We claim that \(\chi(Q_k) = 2\) for all \(k \geq 1\). Indeed, the graph \(Q_k\) is bipartite. We identify the vertex set of \(Q_k\) with

\[
\{(x_1, \ldots, x_k) : x_i = 0 \text{ or } 1 \text{ for all } i\}.
\]

If we let \(A\) be the set of vertices with odd coordinate sum and \(B\) be the set of vertices with even coordinate sum, then \((A, B)\) is a bipartition of the vertex set of \(Q_k\).