Problem 1: Let $G$ be a simple graph with $n$ vertices, each of which has degree $d$. Prove that
\[ \chi(G) \geq \frac{n}{n - d}. \]

Problem 2: Let $G$ be a simple plane graph with no triangles. Use Euler’s Formula to show that $G$ contains a vertex of degree at most 3.

Problem 3: Let $G$ be a simple plane graph with no triangles. Prove that $G$ is 4-colorable. (Use Problem 2.)

Problem 4: A graph $G$ is called $k$-critical if $\chi(G) = k$ and if the deletion of any vertex yields a graph with smaller chromatic number. Find all 2-critical and 3-critical graphs.

Problem 5: Let $G$ be the graph with vertex set given by the states in the United States and having an edge between two states if and only if those states share a border. Compute $\chi(G)$.

Problem 6: Prove that the chromatic polynomial of $K_{2,s}$ is
\[ P_{K_{2,s}}(k) = k(k - 1)^s + k(1)(k - 2)^s. \]

Problem 7: Prove that the chromatic polynomial of $C_n$ is
\[ P_{C_n}(k) = (k - 1)^n + (-1)^n(k - 1). \]

Problem 8: Let $G$ be a disconnected simple graph. Prove that the chromatic polynomial $P_G(k)$ of $G$ is the product of the chromatic polynomials of the components of $G$. What can you say about the degree of the lowest non-vanishing term of $P_G(k)$?