**Math 154: Winter 2017**

**Midterm 1**

**Instructions:** This is a 50 minute exam. No books, notes, electronic devices, or interpersonal assistance are allowed. Write your answers in your blue book, making it clear what problem you are working on. Be sure to justify your answers. This exam is out of 100 points; you get 5 points for legibly writing your name on your blue book. Good luck!

**Problem 1:** [15 pts.] Let \( n \) be a positive integer. Find a closed form for the expression

\[
\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots + \binom{n}{\lfloor n/2 \rfloor + 1},
\]

where \( \lfloor n/2 \rfloor \) is the largest integer \( \leq n/2 \). (Here ‘closed form’ means an expression without a summation sign or a \(+ \cdots +\).)

**Solution:** We claim that this sum equals \( 2^n - 1 \). To see this, let \( S \) be an \( n \)-element set. In homework we proved that there is a bijection between the even and odd subsets of \( S \); in particular, \( S \) has the same number of even subsets as it has odd subsets. The above sum is the number of odd subsets of \( S \). Since \( S \) has a total of \( 2^n \) subsets, we conclude that \( S \) has \( \frac{1}{2}2^n = 2^{n-1} \) odd subsets.

**Problem 2:** [20 pts.] Every point in the plane is colored red, green, or blue. Prove that there is a rectangle in the plane with monochromatic vertices.

**Solution:** Let \( G \) be a \( 4 \times 82 \) grid of points in the plane. Since we have 3 colors, there are \( 3^4 = 81 \) ways to color any column of \( G \). Since we have 82 columns, the Pigeonhole Principle guarantees that there are two columns \( C \neq C' \) which are colored in exactly the same way. Since \( C \) and \( C' \) both have 4 rows and we only have 3 colors, the Pigeonhole Principle again guarantees that two rows of \( C \) and \( C' \) share a color. Taking these two rows in \( C \) and \( C' \) gives us a rectangle with monochromatic vertices.

**Problem 3:** [5 + 5 pts.] Let \( G \) be a graph and let \( C^* \subseteq E(G) \) be a set of edges in \( G \). (a) Carefully define what it means for \( C^* \) to be a ‘cutset’ in \( G \). (b) Give an explicit example of a graph \( G \) and a cutset \( C^* \) in \( G \) containing exactly three edges.

**Solution:** (a) The set \( C^* \) is a cutset if \( G - C^* \) has more components than \( G \) but for any proper subset \( S \subset C^* \) the graph \( G - S \) has the same number of components as \( G \).

(b) Let \( G = K_4 \) be the complete graph on \( \{1, 2, 3, 4\} \). Then \( \{12, 13, 14\} \) is a cutset.

**Problem 4:** [20 pts.] For a positive integer \( n \), let \( b_n \) be the number of subsets \( S \) of \( \{1, 2, \ldots, n\} \) which satisfy the following property:

\[ \text{if } i \in S, \text{ then } i + 1 \notin S \text{ and } i + 2 \notin S \text{ (for all } 1 \leq i \leq n). \]

Find a recursive description of the sequence \( b_n \).
Solution: Let $F_n$ be the collection of subsets $S \subseteq \{1, 2, \ldots, n\}$ which satisfy the given property, so that $b_n = |F_n|$. We have $F_1 = \{\emptyset, \{1\}\}, F_2 = \{\emptyset, \{1\}, \{2\}\},$ and $F_3 = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ so that $b_1 = 2, b_2 = 3,$ and $b_3 = 4.$

For $n \geq 3$ we claim $b_n = b_{n-1} + b_{n-3}$. To see this, fix $n \geq 3$. Define a function

$$\phi: F_n \to F_{n-1} \cup F_{n-3}$$

by

$$\phi(S) = \begin{cases} S \in F_{n-1} & n \notin S \\ S \setminus \{n\} \in F_{n-3} & n \in S. \end{cases}$$

It is easy to see that the function

$$\psi: F_{n-1} \cup F_{n-3} \to F_n$$

given by

$$\psi(T) = \begin{cases} T & T \in F_{n-1} \\ T \cup \{n\} & T \in F_{n-3} \end{cases}$$

is inverse to $\phi$, so that $\phi$ is a bijection and $b_n = b_{n-1} + b_{n-3}$.

Problem 5: [15 pts.] Let $n$ be a positive integer. Prove the following identity:

$$\sum_{k=0}^{n} k(k-1)\binom{n}{k} = n(n-1)2^{n-2}.$$ 

Solution: Consider selecting a committee (of some size) from a class of $n$ students, and then selecting a president and vice president of that committee. Let $a_n$ be the number of ways to do this.

If our committee has size $k$, we have $k(k-1)$ choices for president and vice president. We also have $\binom{n}{k}$ ways to select a size $k$ committee from $n$ students, giving $k(k-1)\binom{n}{k}$ ways to do this if our committee must have size $k$. Summing over all possible committee sizes gives $a_n = \sum_{k=0}^{n} k(k-1)\binom{n}{k}$.

On the other hand, from our class of $n$ students we have $n(n-1)$ ways to choose a president and a vice president. There are $2^{n-2}$ ways to select additional committee members from the remaining $n-2$ students. It follows that $a_n = n(n-1)2^{n-2}$.

Problem 6: [15 pts.] Let $n$ be a positive integer and let $G$ be a simple graph with $n$ vertices and 2 components. What is the maximum number of edges $G$ can have?

Solution: Suppose our components have $a$ and $b$ vertices, where $a+b = n$ and $a, b > 0$. Since we want to maximize the number of edges, we may assume our components are complete graphs, i.e. $G = K_a \cup K_b$. This graph has $\binom{a}{2} + \binom{b}{2}$ edges. We claim that this quantity is maximized when $a = n-1$ and $b = 1$, so the maximum number of edges possible is $\binom{n-1}{2}$. 
To see this, observe that

\[ a(a - 1) + b(b - 1) = a^2 + b^2 - a - b \]

\[ = a^2 + b^2 - n \]

\[ \leq n^2 - n \]

\[ = n(n - 1), \]

where the third line used the fact that \( a^2 + b^2 \leq n^2 \) if \( a, b > 0 \) and \( a + b = n \). Dividing these equations by 2, we see that \( \binom{a}{2} + \binom{b}{2} \leq \binom{n-1}{2} \).