Math 154: Winter 2017
Midterm 2

Instructions: This is a 50 minute exam. No books, notes, electronic devices, or interpersonal assistance are allowed. Write your answers in your blue book, making it clear what problem you are working on. Be sure to justify your answers. This exam is out of 100 points; you get 5 points for legibly writing your name on your blue book. Good luck!

Problem 1: [15 pts.] Draw the complete graph $K_5$ on the torus (i.e. genus 1 orientable surface) in such a way that the edges do not cross.

Solution: (See class notes.) Many students drew figures that were very difficult to read. This is a very difficult problem if you do not use the ‘rectangle’ model of the torus. When you use the rectangle model, be aware that paths need to ‘enter and exit’ the edges at the same point on either side. This was an example from class.

Problem 2: [20 pts.] Let $n > 2$ be an integer. How many labeled trees on the vertex set $\{1, 2, \ldots, n\}$ have both 1 and $n$ as end-vertices (i.e., leaves)?

Solution: Cayley-Prüfer code gives a bijection between trees on the vertex set $1, 2, \ldots, n$ and sequences $(a_1, \ldots, a_{n-2})$ with entries between 1 and $n$. Moreover, the letters appearing in the code of a given tree $T$ are precisely the non-leaves of $T$. We must therefore count sequences $(a_1, \ldots, a_{n-2})$ of length $n-2$ with letters drawn from $\{2, 3, \ldots, n-1\}$. Since we have $n-2$ letters, the number of such sequences is

$$(n - 2)^{n-2}.$$ 

Some students gave the formula $(n - 2)^{n-2}$ with essentially no justification; this was costly. Some other students mentioned ‘Prüfer code’ but did not give its relevant property (that entries correspond to non-leaves). There were also some counting issues here. This was similar to a homework problem (in which only one vertex was a leaf).

Problem 3: [5 + 10 pts.] Let $G$ be a graph. (a) Define what it means for $G$ to be ‘planar’. (b) Give an example of a non-planar graph $G$ which has no subgraph isomorphic to $K_5$ or $K_{3,3}$.

Solution: (a) $G$ is planar if it can be drawn in the plane without edge crossings.

(b) Let $G$ be the graph obtained from $K_5$ by adding a single vertex in the middle of one of its edges. Then $G$ is non-planar (it is homeomorphic to $K_5$) but no subgraph of $G$ is isomorphic to $K_5$ or $K_{3,3}$.

There were significant issues with Part (b) – many students did not understand the ‘subgraph’ condition. Also, many students wrote down graphs which were planar.

Problem 4: [15 pts.] Let $G$ be a Hamiltonian graph and let $S$ be a set of $k$ vertices in $G$. Prove that $G - S$ has at most $k$ components.
Solution: Let \( C \) be a Hamiltonian cycle of \( G \). Then \( C \) contains all of the vertices of \( G \), so that \( C - S \) has at most \( k \) nonempty components. Since the components of \( G - S \) are unions of components of \( C - S \), we see that \( G - S \) has at most \( k \) components.

Many students assumed that \( G \) was a cycle, or did not relate the components of \( C - S \) with the components of \( G - S \). This was a homework problem.

Problem 5: [15 pts.] Find a planar graph \( G \) with chromatic number \( \chi(G) = 4 \) such that no subgraph of \( G \) is isomorphic to the complete graph \( K_4 \).

Solution: Let \( G = W_6 \) be the ‘wheel’ graph on 6 vertices. Since \( \chi(C_3) = 3 \) (odd cycle) and the center vertex of \( W_6 \) must get a different color from its edges, we have \( \chi(W_6) = 4 \). Clearly \( W_6 \) is planar and has no subgraph isomorphic to \( K_4 \).

Most students did not receive credit for this problem. There were many graphs that were 2-colorable or 3-colorable. Colorability vs. planarity generated a lot of confusion. I gave the wheel example in lecture.

Problem 6: [15 pts.] Give an example of two simple graphs \( G \) and \( H \) which have the same number of vertices and edges but different chromatic polynomials.

Solution: Let \( G = K_3 \cup K_1 \cup K_1 \cup K_1 \) be the union of a 3-cycle and three vertices and let \( H = K_2 \cup K_2 \cup K_2 \) be a union of three edges. Then \( G \) and \( H \) have the same number of vertices (6) and edges (3).

We know from homework that the chromatic polynomial of a graph is the product of the chromatic polynomials of its components. Moreover, we have \( P_{K_1}(k) = k, P_{K_2}(k) = k(k-1), \) and \( P_{K_3}(k) = k(k-1)(k-2) \). It follows that

\[
P_G(k) = k^3(k-1)(k-2)
\]

and

\[
P_H(k) = k^3(k-1)^3,
\]

so that \( P_G(k) \neq P_H(k) \).

This was an example from class. Students got some credit for getting the example correct, some more credit for getting the chromatic polynomials right, and some more credit for explaining how they got the chromatic polynomials.