Math 184A: Fall 2013
Homework 5
Due 5:00pm on Friday 11/8/2013

Problem 1: (Exercise 6.40 in Bóna) Let n be divisible by 3. Prove that $c(n, n/3) \geq \frac{n!}{3^{n/3}(n/3)!}$.

Problem 2: (Exercise 6.41 in Bóna) Prove that for all positive integers n, r, and k such that $n = rk$, the inequalities

$$(r-1)!^k \leq \frac{c(n,k)}{S(n,k)} \leq (n-k)!$$

hold.

Problem 3: (Exercise 6.43 in Bóna) Let $a(n,k)$ be the number of permutations in S_n with k cycles in which the entries 1 and 2 are in the same cycle. Prove that for $n \geq 2$, we have

$$\sum_{k=1}^{n} a(n,k)x^k = x(x+2)(x+3)\cdots(x+n-1).$$

Problem 4: (Exercise 6.46 in Bóna) A group of n tourists arrives at a restaurant. They sit down around circular tables, leaving no table empty. Then each table orders one of r possible drinks. Prove that the number of ways this can happen is $r(r+1)\cdots(n+r-1)$. (Two seating arrangements are considered the same if each person has the same person to their left.)

Problem 5: (Exercise 6.47 in Bóna) We write each element of $[n-1]$ on a separate card, then randomly select any number of cards, and take the product of the numbers written on them. Then we do this for all 2^{n-1} possible subsets of the set of $n-1$ cards. (The empty product is taken to be 1.) Finally, we take the sum of the 2^{n-1} products we obtained. What is this sum?

Problem 6: (Exercise 6.52 in Bóna) Recall the definition of the inversions of a permutation from Homework 4. Let $I(n,k)$ be the number of permutations in S_n that have k inversions. Prove that $I(n,k) = I(n, \binom{n}{2} - k)$.

Problem 7: (Exercise 6.53 in Bóna) Let $I(n,k)$ be defined as in the previous exercise. Prove that

$$\sum_{k=0}^{\binom{n}{2}} I(n,k)x^k = (1+x)(1+x+x^2)\cdots(1+x+\cdots+x^{n-1}).$$

Problem 8: (Exercise 6.54 in Bóna) In Homework 4 you gave a combinatorial proof of the fact that there are $\frac{n!}{2}$ even permutations in S_n. Use Problem 7 of this homework to give another proof of this fact.