### Last Time

The number of permutations \( \pi \in S_n \) of cycle type \( (a_1, \ldots, a_n) \) is given by:

\[
\frac{n!}{(a_1! \cdot a_2! \cdots a_n!)}
\]

where \( a_i \) are the cycle lengths.

The signless Stirling number of the first kind \( c(n, k) \) counts the number of permutations in \( S_n \) with exactly \( k \) cycles.

The Stirling number of the first kind \( s(n, k) \) is related to \( c(n, k) \) by:

\[
s(n, k) = (-1)^{n-k} c(n, k).
\]

### Recursion

Let \( n, k > 0 \).

Recursion:\

* \( c(n, k) = c(n-1, k) + (n-1) \cdot c(n-1, k) \) *

### Conventions

- \( c(n, 0) = 1 \) for \( n = 0 \)
- \( c(n, n) = 1 \) for all \( n \)
- \( c(n, k) = 0 \) if \( n < k \)

### Proposition

The sum of the Stirling numbers of the first kind is given by:

\[
\sum_{k=0}^{n} c(n, k) \cdot x^k = x \cdot (x+1)(x+2) \cdots (x+n-1).
\]

### Example (\( n = 4 \))

\[
x(x+1)(x+2)(x+3) = c(4,0) + c(4,1)x + c(4,2)x^2
\]

\[
+ c(4,3)x^3 + c(4,4)x^4
\]

\[
= 0 + 6x + 11x^2 + 6x^3 + x^4.
\]

### Remark

Could also write:

\[
\sum_{k=0}^{\infty} c(n, k) x^k = \sum_{k>0} c(n, k) x^k = x(x+1) \cdots (x+n-1).
\]
Pf We have \( F_n(x) = \sum_{k \geq 0} c(n,k) \cdot x^k \).

Starting with

\[
\circ \quad c(n,k) = c(n-1,k) + (n-1) c(n-1,k-1)
\]

multiply both sides by \( x^k \) to get

\[
\circ \circ \quad c(n,k) x^k = c(n-1,k) x^k + (n-1) c(n-1,k-1) x^k.
\]

Now sum over \( k \geq 0 \) to get

\[
\circ \circ \circ \quad \sum_{k \geq 0} c(n,k) x^k = \sum_{k \geq 0} c(n-1,k) x^k + (n-1) \sum_{k \geq 0} c(n-1,k-1) x^k,
\]

or

\[
\circ \quad F_n(x) = x F_{n-1}(x) + (n-1) \cdot F_{n-1}(x)
\]

\[
= (x + n-1) F_{n-1}(x).
\]

Since \( F_1(x) = \frac{x}{2} \), we have

\[
F_n(x) = x(x+1) - (x+n-1).
\]
Application \[ \ln \sum_{k=0}^{n} c(n,k) x^k = x(x+1) \cdots (x+n-1), \]

set \( x \mapsto -x \) & multiply by \( (-1)^n \):

\[ \sum_{k=0}^{n} s(n,k) x^k = x \cdot (x-1) \cdots (x-n+1) = (x)_n. \]

However, \[ \sum_{k=0}^{n} S(n,k)(x)_k = x^n, \]

**Q** How are \( S(n,k) \) and \( s(n,k) \) related?

Consider the infinite matrices:

\[ S = (S(n,k))_{n,k \geq 0} = \begin{bmatrix} S(0,0) & S(0,1) & S(0,2) & \cdots \\ S(1,0) & S(1,1) & S(1,2) & \cdots \\ S(2,0) & S(2,1) & S(2,2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \]

\[ s = (s(n,k))_{n,k \geq 0} = \begin{bmatrix} s(0,0) & s(0,1) & s(0,2) & \cdots \\ s(1,0) & s(1,1) & s(1,2) & \cdots \\ s(2,0) & s(2,1) & s(2,2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \]

(both lower \( \Delta \)-lau)

Theorem \( sS = Ss = I \) (matrix multiplication).
Pf. Let $V = \left\{ \frac{\text{all polynomials}}{a_nx^n + \ldots + a_1x + a_0: a_i \in \mathbb{R}} \right\} \ (\mathbb{R}[x])$.

Then $V$ is an infinite-dimensional $\mathbb{R}$-vector space. $V$ has two different bases:

$B = \{1, x, x^2, x^3, \ldots\}$

$E = \{1, (x), (x)_2, (x)_3, \ldots\}$

The matrices $S$ and $s$ are transition matrices between these bases.

§ 7 Principle of Inclusion-Exclusion

Ex. 100 students take math
150 physics.

Q. How many take math or physics?
A. Not enough information.

If 50 take math AND physics,

\[ \begin{align*}
100 & \quad \text{taking math} \\
150 & \quad \text{taking physics} \\
180 & \quad \text{taking either or both} \\
50 & \quad \text{taking both} \\
\hline
100 + 150 - 50 & = 200.
\end{align*} \]