Math 184 A Lecture 12 10/27/2017

Last Time

$$\#\text{ of perms } \pi \in S_n \text{ of cycle type } (a_1, \dots, a_n) = n! \Big/ \left(1^{a_1} 2^{a_2} \cdots n^{a_n}\, a_1! \cdot a_2! \cdots a_n!\right).$$

$c(n,k)$ = signless Stirling # of first kind = # of $\pi \in S_n$ with k cycles

$s(n,k)$ = Stirling # of first kind $= (-1)^{n-k} c(n,k)$.

Recursion Let $n, k > 0$. We have

$$* \quad c(n,k) = c(n-1, k-1) + (n-1) \cdot c(n-1, k). \; *$$

Conventions

- $c(n,0) = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}$
- $c(n,k) = 0$ if $n < k$.

Proposition

$$\sum_{k=0}^{n} c(n,k) \cdot x^k = x \cdot (x+1)(x+2) \cdots (x+n-1).$$

Ex $(n = 4)$

$$x(x+1)(x+2)(x+3) \overset{?}{=} c(4,0) + c(4,1)x + c(4,2)x^2 + c(4,3)x^3 + c(4,4)x^4$$

$$= 0 + 6x + 11x^2 + 6x^3 + x^4. \checkmark$$

Rmk Could also write

$$\sum_{k=0}^{\infty} c(n,k) x^k = \sum_{k \geq 0} c(n,k) x^k = x(x+1) \cdots (x+n-1).$$

Pf We let $F_n(x) = \sum_{k \geq 0} c(n,k) \cdot x^k$.

Starting with

$$(*) \quad c(n,k) = c(n-1,k-1) + (n-1)\, c(n-1,k)$$

multiply both sides by x^k to get

$$(**) \quad c(n,k)\, x^k = c(n-1,k-1)\, x^k + (n-1)\, c(n-1,k)\, x^k.$$

Now sum over $k \geq 0$ to get

$$(***) \quad \sum_{k \geq 0} c(n,k)\, x^k = \sum_{k \geq 0} c(n-1,k-1)\, x^k + (n-1) \sum_{k \geq 0} c(n-1,k)\, x^k,$$

or

$$(\dagger) \quad F_n(x) = x F_{n-1}(x) + (n-1) \cdot F_{n-1}(x)$$

$$= (x + n - 1)\, F_{n-1}(x).$$

Since $F_1(x) = x$, we have

$$F_n(x) = x(x+1) \cdots (x+n-1).$$

<u>Application</u>? In $\displaystyle\sum_{k=0}^{n} c(n,k)\, x^k = x(x+1)\cdots(x+n-1)$,

set $x \longmapsto -x$ & multiply by $(-1)^n$,

$$\boxed{\sum_{k=0}^{n} s(n,k)\, x^k = x\cdot(x-1)\cdots(x-n+1) = (x)_n .}$$

However, $\boxed{\displaystyle\sum_{k=0}^{n} S(n,k)\,(x)_k = x^n}$

<u>Q</u> How are $S(n,k)$ and $s(n,k)$ <u>related</u>?

Consider the <u>Infinite</u> <u>Matrices</u>

$$S = \big(S(n,k)\big)_{n,k\geq 0} = \begin{bmatrix} S(0,0) & \cancelto{0}{S(0,1)} & \cancelto{0}{S(0,2)} & 0 \\ S(1,0) & S(1,1) & \cancelto{0}{S(1,2)} & \\ S(2,0) & S(2,1) & S(2,2) & \\ & & & \ddots \end{bmatrix}$$

$$s = \big(s(n,k)\big)_{n,k\geq 0} = \begin{bmatrix} s(0,0) & \cancelto{0}{s(0,1)} & \cancelto{0}{s(0,2)} & 0 \\ s(1,0) & s(1,1) & \cancelto{0}{s(1,2)} & \\ s(2,0) & s(2,1) & s(2,2) & \\ & & & \ddots \end{bmatrix}$$

(both lower Δ'lar)

<u>Theorem</u> $\quad sS = Ss = I$ (<u>matrix</u> <u>multiplication</u>).

Pf Let $V = \left\{ \begin{array}{l} \text{all polynomials} \\ a_n x^n + \cdots + a_1 x + a_0 \end{array} : a_i \in \mathbb{R} \right\}$ $(\mathbb{R}[x])$.

Then V is an infinite-dim'l $\mathbb{R}$-vector space. V has two different bases:

$$\mathcal{B} = \{1, x, x^2, x^3, \ldots\}$$

$$\mathcal{C} = \{1, (x)_1, (x)_2, (x)_3, \ldots\}$$

The matrices S and s are transition matrices btwn these bases.

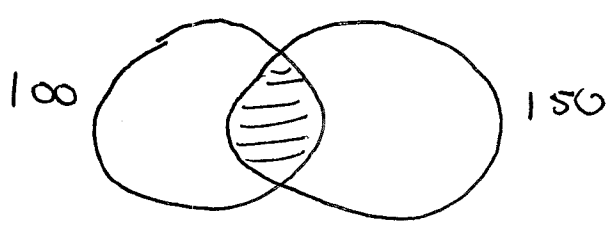
§7 Principle of Inclusion - Exclusion

Ex 100 students take math
150 ——"—— physics.

Q How many take math OR physics?

A Not enough information!

If 50 take math AND physics,



taking math OR physics $= 100 + 150 - 50$
$= 200.$