Math 184A Lecture 4 10/6/2017

Last Time

Given a set X, find $|X|$.

Bijective Method - Find a set Y where we know $|Y|$.

- Find a bij'n $f: X \to Y$.
- Then $|X| = |Y|$.

Ex If $n \geq 0$, then $|\{(A,B) : A \subseteq B \subseteq [n]\}| = 3^n$.

- $\binom{n}{k}$ = # of k-elt subsets of an n-elt set "binomial coeff."
- If $0 \leq k \leq n$ then $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
- $\binom{n}{k} = 0$ if $k > n$.

Q A given quarter has 40 class days. We must give 5 exams, but no two can be < 3 days apart. (eg exam no ex. no ex. exam ok!)
How many ways to schedule exams?

A Let $\{1, \ldots, 40\}$ be the class days & $1 \leq a_1 < \cdots < a_5 \leq 40$ be the exam days. Then $a_1 < a_2 - 2 < a_3 - 4 < a_4 - 6 < a_5 - 8$. The fcn

$$\varphi : \left\{ \begin{array}{c} \text{valid exam placements} \\ a_1 < a_2 < a_3 < a_4 < a_5 \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{5-elt subsets} \\ \text{of } [32] \end{array} \right\}$$

$$a_1 < a_2 < a_3 < a_4 < a_5 \longmapsto \{a_1 < a_2 - 2 < a_3 - 4 < a_4 - 6 < a_5 - 8\}$$

is a bijection (indeed, its inverse is

ψ: $\{$5-elt subsets of $[32]\} \longrightarrow \{$valid exam placements$\}$

$\{b_1 < b_2 < b_3 < b_4 < b_5\} \longrightarrow \{b_1 < b_2+2 < b_3+4 < b_4+6 < b_5+8\}$.)

Thus

$$\#\text{ of valid exam placements} = \#\text{ of 5-elt subsets of } [32] = \binom{32}{5}.$$

Q We have 4 flavors of ice cream (1, 2, 3, 4) & must buy 3 pints. In how many ways can this be done?

111	122	134	224	333
112	123	144	233	334
113	124	222	234	344
114	133	223	244	444

$\Rightarrow 20$.

Think 133 $\longleftrightarrow$ * | | * * | "stars & bars"

1 fl. 1 0 fl. 2 2 fl. 3 0 fl. 4

134 $\longleftrightarrow$ * | | * | *

$\Rightarrow \binom{6}{3}$ choose 3 *'s.

Thm The number of k-elt multisets with elts drawn from $[n]$ is $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$.

eg $n = 4$, $k = 7$ 1113344 $\longleftrightarrow$ * * * | | * * | * *

$n+k-1 = 10$ things

$k = 7$ *'s, $n-1 = 3$ |'s.

§4 The Binomial Theorem

Binomial Theorem Let x, y be commuting variables ($xy = yx$). If $n \in \mathbb{Z}_{\geq 0}$ then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Pf We have $(x+y)^n = \overbrace{(x+y)(x+y) \cdots (x+y)}^{n}$.

The coef. of $x^k y^{n-k}$ is the # of ways to choose x from k of the factors (and so y from the $n-k$ remaining factors). This coef. is $\binom{n}{k}$. //

Prop Let $n > 0$. Then $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Pf In the Binomial Thm

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad \text{set} \quad x = -1 \text{ and } y = 1:$$

$$0 = 0^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}. \quad //$$

eg $\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 1 - 4 + 6 - 4 + 1 = 0$ ✓

Pascal Recursion Let $n, k > 0$. We have

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

"Pascal's Δ"

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\vdots$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1.$$
$$\vdots$$

"Algebraic" Pf.

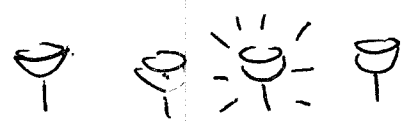
We have

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!\cdot(n-k-1)!} + \frac{(n-1)!}{(k-1)!\,(n-k)!}$$

$$= \frac{(n-1)!\cdot(n-k) + (n-1)!\cdot k}{k!\cdot(n-k)!}$$

$$= \frac{(n-1)!\cdot n}{k!\cdot(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}. \quad //$$

* Correct, but doesn't give much insight.

"Combinatorial Pf" 

Let S be a set of n grails, with one Holy Grail.

Then

$\binom{n}{k}$ = # of ways to choose k grails from S

= # of ways to choose k grails from S AND choose HG + # of ways to choose k grails from S and NOT choose HG

$$= \binom{n-1}{k-1} + \binom{n-1}{k}. \quad //$$

* To prove $A = B$ combinatorially, prove A and B count the same set in two different ways. (More insight.)