Math 184A  Lecture 4  10/6/2017

Last Time

Given a set $X$, find $|X|$.  

**Bijective Method** - Find a set $Y$ where we know $|Y|$.  

1. Find a bijection $f : X \rightarrow Y$.  
2. Then $|X| = |Y|$.  

**Example**  
If $n > 0$, then $\left| \{(A, B) : A \leq B \in [n] \} \right| = 3^n$.  

$\binom{n}{k}$ = \# of $k$-elt subsets of an $n$-elt set "binomial coeff."  

- If $0 \leq k \leq n$ then $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.  
- $\binom{n}{k} = 0$ if $k > n$.  

Q: A given quarter has 40 class days. We must give 5 exams, but no two can be < 3 days apart. (e.g. exam 10:00, exam 10:30, exam 11:30)  

How many ways to schedule exams?

Let $\{1, \ldots, 40\}$ be the class days & $1 \leq a_1 < \ldots < a_5 \leq 40$ be the exam days. Then $a_1 < a_2 - 2 < a_3 - 4 < a_4 - 6 < a_5 - 8$.  

The fcn $Q : \{\text{valid exam placements}\} \rightarrow \{\text{5-elt subsets of } [32]\}$  

$a_1 < a_2 < a_3 < a_4 < a_5 \mapsto \{a_1, a_2 - 2, a_3 - 4, a_4 - 6, a_5 - 8\}$ is a bijection (indeed, its inverse is
\[ \Psi: \{5\text{-elt subsets of } [32]\} \rightarrow \{\text{valid exam placements}\} \]
\[ \{b_1 < b_2 < b_3 < b_4 < b_5 \} \rightarrow \{b_1 < b_2 + 2 < b_3 + 4 < b_4 + 6 < b_5 + 8\} \]

Thus

\[ \# \text{ of valid exam placements} = \# \text{ of } 5\text{-elt subsets of } [32] = \binom{32}{5} \]

\[ (1,2,3,4) \]

Q: We have 4 flavors of ice cream & must buy 3 pints. In how many ways can this be done?

\[
\begin{array}{cccccc}
111 & 122 & 134 & 224 & 333 & \Rightarrow 20 \\
112 & 123 & 144 & 233 & 334 & \\
113 & 124 & 222 & 234 & 344 & \\
114 & 133 & 223 & 244 & 444 & \\
\end{array}
\]

Think 133 \[ \leftrightarrow \] * | | | * * | "stars & bars"

\[
\begin{array}{cccc}
1 \text{ fl.1} & 0 \text{ fl.2} & 2 \text{ fl.3} & 0 \text{ fl.1} \\
\end{array}
\]

134 \[ \leftrightarrow \] * | | | * * *

\[ \Rightarrow \binom{\binom{3}{2}}{3} \text{ choose 3 *'s} \]

The number of k-elt multisets with elt's drawn from [n] is \( \binom{n+k-1}{k} = \binom{n+k-1}{n-1} \).

cg \ n=4 \quad 1113344 \leftrightarrow * * * \mid | * * | * * * \mid * *

n+k-1 = 10 \text{ things}

k = 7 \quad *'s, \quad n-1 = 3 \mid 5
§4 The Binomial Theorem

Binomial Theorem. Let \( x, y \) be commuting variables (\( xy = yx \)).

If \( n \in \mathbb{N} \geq 0 \) then
\[
(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\]

\[\text{Pf.}\] We have \( (x+y)^n = (x+y)(x+y)^{n-1} \).

The coef. of \( x^k y^{n-k} \) is the \# of ways to choose \( x \) from \( k \) of the factors (and so \( y \) from the \( n-k \) remaining factors). This coef. is \( \binom{n}{k} \).

Prop. Let \( n > 0 \). Then \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \).

\[\text{Pf.}\] In the Binomial Thm.
\[
(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad \text{set} \quad x = -1 \text{ and } y = 1:
\]
\[
0 = 0^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}. \quad \checkmark
\]

\[\text{eg.}\] \( \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 - 3 + 3 - 1 = 0 \).

Pascal Recursion. Let \( n > 0 \). We have
\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.
\]

"Pascal's \( \Delta^n \)"
\[
\begin{array}{cccccc}
\binom{0}{0} & \binom{0}{1} & \binom{0}{2} & \binom{0}{3} & \binom{0}{4} & \binom{0}{5} \\
\binom{1}{0} & \binom{1}{1} & \binom{1}{2} & \binom{1}{3} & \binom{1}{4} & \binom{1}{5} \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \binom{2}{3} & \binom{2}{4} & \binom{2}{5} \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \binom{3}{4} & \binom{3}{5} \\
\binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} & \binom{4}{5} \\
\binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5}
\end{array}
\]
"Algebraic Pf."

We have
\[
\binom{n-1}{k} \neq \binom{n-1}{k-1} = \frac{(n-1)!}{k! \cdot (n-k-1)!} + \frac{(n-1)!}{(k-1)! \cdot (n-k)!}
\]

\[
= \frac{(n-1)! \cdot (n-k) + (n-1)! \cdot k}{k! \cdot (n-k)!}
\]

\[
= \frac{(n-1)! \cdot n}{k! \cdot (n-k)!} = \frac{n!}{k! \cdot (n-k)!} = \binom{n}{k}.
\]

* Correct, but doesn't give much insight.

"Combinatorial Pf."

Let \( S \) be a set of \( n \) grails, with one Holy Grail.

Then
\[
\binom{n}{k} = \text{# of ways to choose } k \text{ grails from } S
\]

\[
= \text{# of ways to choose } k \text{ grails from } S \text{ and } \text{# of ways to choose } k \text{ grails from } S \text{ and not choose HG}
\]

\[
= \binom{n-1}{k-1} + \binom{n-1}{k}.
\]

* To prove \( A = B \) combinatorially, prove \( A \) and \( B \) count the same set in two different ways.

(More insight.)