Math 184A  Lecture 8  10/16/2017

Last Time * Set pts.

\[ B(n) = \text{Bell number} = \#	ext{ of ptms of } [n] \]
\[ S(n,k) = \text{Stirling # of } \text{ the 2nd kind} = \#	ext{ of ptms of } [n] \text{ into } k \text{ blocks} \]

Recursion \[ S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k). \]

Fact We have \[ B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(n-i). \]

Pf Given any set ptm \( T \) of \( [n+1] \), erasing the blk \( B_0 \) of \( T \) containing \( n+1 \) gives a set ptm of a set of size \( n+1 - |B_0| \). This leads to \[ B(n+1) = \sum_{S \subseteq [n]} B(n-1|S|) = \sum_{i=0}^{n} \binom{n}{i} B(n-i). \]

Application The number of surjective fnses \( f: [n] \to [k] \)
is \[ k! \cdot \text{Stir}(n,k). \]

Pf \[ \begin{array}{c}
[\sigma(1)] \quad 1

[\sigma(2)] \quad 2

[\sigma(k)] \quad k
\end{array} \]

A surj \( f: [n] \to [k] \) is determined by a k-blk set ptm \( \{B_1, \ldots, B_k\} \)
of \( [n] \), and an permutation of \( \{B_1, \ldots, B_k\} \).

This gives \( \text{Stir}(n,k) \cdot k! \). Adios for \( \sigma \).
Application Let $x$ be a variable. For $k > 0$ let

$$(x)_k \equiv x \cdot (x-1) \cdots (x-k+1).$$

We have $\bigcirc \quad x^n = \sum_{k=0}^{n} S(n,k) \cdot (x)_k.$

$$\begin{array}{ll}
E_{x} (n=3) & \quad 1 \\
x^3 & = S_0^{(3,0)} \cdot x^3 + S_1^{(3,1)} \cdot x + S_2^{(3,2)} \cdot x \cdot (x-1) \\
& \quad + S_3^{(3,3)} \cdot x \cdot (x-1) \cdot (x-2) \\
& = 3 \cdot x^3 - 3x^2 + x^3 - 3x^2 + 2x = x^3.
\end{array}$$

Since $\bigcirc$ is a degree $n$ polynomial identity, it suffices to check that $\bigcirc$ is true for all positive integers $x$. Indeed, if $x \in \mathbb{N}_{\geq 0},$

$$x^n = \# \left\{ \text{functions } f : [n] \rightarrow [x] \right\}$$

$$= \sum_{T \subseteq [x]} \# \left\{ f : [n] \rightarrow [x] : \text{magee}(f) = T \right\}$$

$$= \sum_{T \subseteq [x]} |T| ! \cdot S_0^{(n,|T|)} = \sum_{k=0}^{n} S(n,k) = \sum_{k=0}^{n} \binom{x}{k} \cdot k! \cdot S(n,k) \quad \text{as desired.}$$
Integer Partitions

Let \( n \in \mathbb{Z}^+ \)

Def: A sequence \( (a_1, \ldots, a_k) \) of positive ints is a partition of \( n \) if \( a_1 + \ldots + a_k = n \).

Ex: partitions of 6

\( 6, 51, 42, 33, 411, 321, 222, 2211, 3111, 21111, 111111 \)

* Partitions may be represented by Ferrers diagrams

\[ (4,4,2,1) \leftrightarrow \begin{array}{c}
4 \\
4 \\
2 \\
1 \\
\end{array} \]

Def: The partition number is \( p(n) = \# \text{ of (integer) partitions of } n \).

Also, \( \pi_k(n) = \# \text{ of partitions of } n \text{ into } k \text{ parts.} \)

\( \begin{align*}
\pi_1(6) &= 1 \\
\pi_2(6) &= 3 \\
\pi_3(6) &= 1 \\
\pi_4(6) &= 1 \\
\pi_5(6) &= 1 \\
\pi_6(6) &= 1
\end{align*} \)
Def Let $\lambda$ be a partition of $n$. The conjugate $\lambda'$ of $\lambda$ is obtained by reflecting the Ferrers shape of $\lambda$ across its main diagonal.

$\lambda = \begin{array}{ccc}
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{array}$

$\lambda' = \begin{array}{ccc}
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{array}$

(4 2 1) $\quad$ (3 2 1 1)

Obs The map $\{\text{partitions of } n\} \rightarrow \{\text{partitions of } n\}$

$\lambda \mapsto \lambda'$

is a bijection (and an involution) $(\lambda')' = \lambda$.

Fact For any $n > 0$, $k > 0$:

\# of partitions of $n$ with $k$ parts = \# of partitions of $n$ whose largest part has size $k$.

($\pi_k(n)$)

Why? \# of partitions of $\lambda$ = size of largest part of $\lambda'$, for any ptn $\lambda$.  
