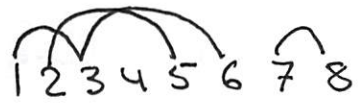
Math 184A — Lecture 8 — 8/16/2017

Last Time * Set ptns. $\{136/25/4/78\}$

$B(n)$ = Bell number = # of ptns of $[n]$

$S(n,k)$ = Stirling # of the 2nd kind = # of ptns of $[n]$ into k blocks



"arc diagram"

Recursion $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.

Fact We have $B(n+1) = \sum_{i=0}^{n} \binom{n}{i} \cdot B(n-i)$.

Pf Given any set ptn π of $[n+1]$, erasing the blk B_0 of π containing $n+1$ gives a set ptn of a set of size $n+1-|B_0|$.

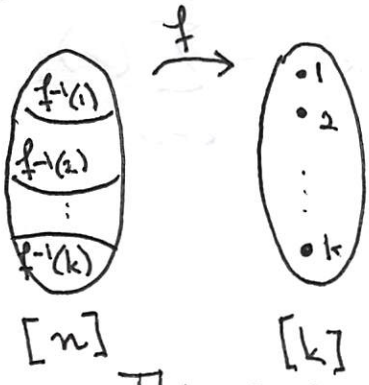
This leads to

$$B(n+1) = \sum_{\substack{S \subseteq [n] \\ S = B_0 - \{n+1\}}} B(n-|S|) = \sum_{i=0}^{n} \binom{n}{i} B(n-i).$$

//

Application The number of surjective fcns $f: [n] \to [k]$ is $k! \cdot \mathrm{Stir}(n,k)$.

Pf



A surj'n $f: [n] \to [k]$ is determined by

- a k-blk set ptn $\{B_1, \ldots, B_k\}$ of $[n]$, and
- a permutation of $\{B_1, \ldots, B_k\}$ determining the images of $B_1, \ldots, B_k$.

This gives $\mathrm{Stir}(n,k) \cdot k!$ choices for f. //

<u>Application</u> Let x be a <u>variable</u>. For $k \geq 0$ let

$$(x)_k \equiv x \cdot (x-1) \cdot \ldots \cdot (x-k+1).$$

We have $$(*) \quad x^n = \sum_{k=0}^{n} S(n,k) \cdot (x)_k.$$

<u>Ex</u> $(n=3)$

$$x^3 \overset{?}{=} \cancelto{0}{S(3,0)}\cancelto{1}{(x)_0} + \cancelto{1}{S(3,1)} \cdot x + \cancelto{3}{S(3,2)} \cdot x \cdot (x-1)$$

$$+ \cancelto{1}{S(3,3)}\, x \cdot (x-1) \cdot (x-2)$$

$$= x + 3x^2 - 3x + x^3 - 3x^2 + 2x = x^3. \checkmark$$

<u>Pf</u> Since $(*)$ is a degree n polynomial identity, it suffices to check that $(*)$ is true for all positive integers x. Indeed, if $x \in \mathbb{Z}_{>0}$,

$$x^n = \#\left\{\text{functions } f : [n] \longrightarrow [x]\right\}$$

$$= \sum_{T \subseteq [x]} \#\left\{ f : [n] \to [x] : \text{image}(f) = T \right\}$$

$$= \sum_{T \subseteq [x]} |T|! \cdot S(n, |T|)$$

$$= \sum_{k=0}^{n} \binom{x}{k} \cdot k! \cdot S(n,k) = \sum_{k=0}^{n} (x)_k \cdot S(n,k),$$

<u>as desired</u>. //

Integer Partitions

Let $n \in \mathbb{Z}_{>0}$

Def A sequence $(a_1 \geq \cdots \geq a_k)$ of positive ints is a (weakly decreasing) <u>partition of n</u> if $a_1 + \cdots + a_k = n$.

Ex <u>partitions of 6</u>

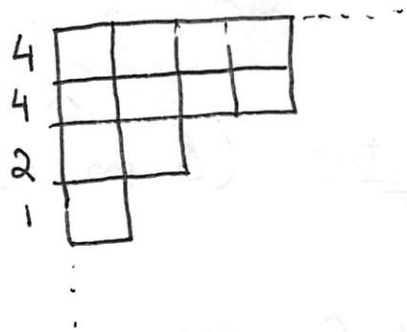
6, 51, 42, 33, 411, 321, 222, 2211, 3111, 21111, 111111

* Partitions may be represented by <u>Ferrers diagrams</u>

eg

$(4,4,2,1) \longleftrightarrow$

ptn of 11



Def The <u>partition number</u> is $p(n) = \#$ of (integer) ptns of n.

Also, $p_k(n) :=$ # of ptns of n <u>into k parts</u>.

eg

$p_1(6) = 1$

$p_2(6) = 3$

$p_3(6) = 3$

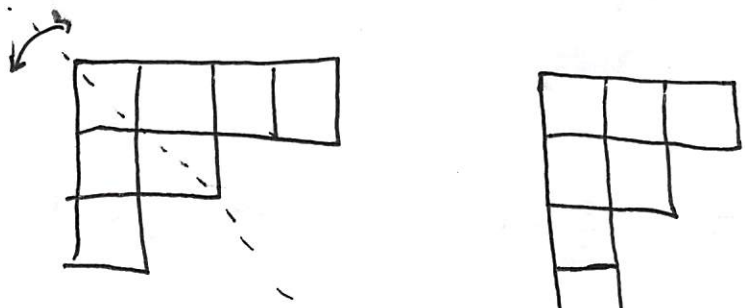
$p_4(6) = 2$

$p_5(6) = 1$

$p_6(6) = 1$

$\Rightarrow$ $\underline{p(6) = 11}$

Def Let λ be a partition of n. The <u>conjugate</u> λ' of λ is obtained by reflecting the Ferrers shape of λ across its main diagonal.

$\lambda =$ 

$(4\ 2\ 1)$ $\qquad$ $= \lambda'$

$(3\ 2\ 1\ 1)$

Obs The map $\{\text{partitions of } n\} \longrightarrow \{\text{ptns of } n\}$

$$\lambda \longmapsto \lambda'$$

is a <u>bijection</u> (& an <u>involution</u>). $(\lambda')' = \lambda$.

Fact For any $n > 0$, $k > 0$:

$$\begin{pmatrix} \#\text{ of ptns of } n \\ \text{with } k \text{ parts} \\ \| \\ P_k(n) \end{pmatrix} = \begin{pmatrix} \#\text{ of ptns of } n \\ \text{whose largest part} \\ \text{has size } k. \end{pmatrix}$$

Why? $\#$ of parts of λ = size of largest part of λ', for <u>any</u> ptn λ.