Math 190: Fall 2014
Homework 6
Due 5:00pm on Friday 11/21/2014

Problem 1: (Problem 24.1 in Munkres) (a) Show that no two of $(0, 1), (0, 1]$, and $[0, 1]$ are homeomorphic. (b) Suppose there exist imbeddings $f : X \to Y$ and $g : Y \to X$. Give an example to show that X and Y need not be homeomorphic. (c) Show that $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic for $n > 1$. (In fact, $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$. This is harder to show.)

Problem 2: (Exercise 24.2 in Munkres) Let $f : S^1 \to \mathbb{R}$ be a continuous function. Show there exists a point $x \in S^1$ such that $f(x) = f(-x)$.

Problem 3: (Exercise 24.4 in Munkres) Let $(X, <)$ be an ordered set which is connected in the order topology. Prove that X is a linear continuum.

Problem 4: (Exercise 24.8 in Munkres) (a) Is a product of path connected spaces necessarily path connected? (b) Let $A \subset X$ and assume that A is path connected. Is $\bar{A}$ necessarily path connected? (c) Let $f : X \to Y$ be a surjective continuous map and assume that X is path connected. Is Y necessarily path connected? (d) Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of path connected subspaces of a space X with $\bigcap_{\alpha \in J} A_\alpha$ nonempty. Is $\bigcup_{\alpha \in J} A_\alpha$ necessarily path connected?

Problem 5: (Exercise 25.4 in Munkres) Suppose X is locally path connected. Show that every connected open set in X is path connected.

Problem 6: (Exercise 25.1 in Munkres) What are the components and path components of $\mathbb{R}_\ell$? What are the continuous maps $f : \mathbb{R} \to \mathbb{R}_\ell$?

Problem 7: (Exercise 26.3 in Munkres) Prove that a finite union of compact subspaces of X is compact.

Problem 8: (Exercise 26.4 in Munkres) Let (X, d) be a metric space. A subset $S \subset X$ is called *bounded* if there exists $M \geq 0$ such that $d(x, y) \leq M$ for all $x, y \in S$. Prove that every compact subspace of X is closed and bounded. Prove that a closed and bounded subset of X need not be compact.

Problem 9: Let $S^n \subset \mathbb{R}^{n+1}$ denote the n-dimensional sphere. *(n-dimensional, real) Projective space* P^n is defined to be the quotient $P^n := S^n / \sim$, where we declare that $x \sim -x$ for all $x \in S^n$. Consider the n-dimensional unit disk D^n and its boundary $S^{n-1} \subset D^n$. Let $R^n := D^n / \sim_0$, where we identify $y \sim_0 -y$ for all $y \in S^{n-1}$. Prove that P^n is homeomorphic to R^n.