Math 190: Fall 2014
Midterm 1
Solutions and Comments

Instructions: This is a 50 minute exam. No books, notes, electronic devices, or interpersonal assistance are allowed. Write your answers in your blue book, making it clear what problem you are working on. Be sure to justify your answers. This exam is out of 115 points; you get 5 points for legibly writing your name on your blue book. Good luck!

Problem 1: [5 + 15 points] (a) Carefully define the "lower limit topology" $\mathbb{R}_\ell$ on the set of real numbers. (b) To what point or points (if any) does the sequence $x_n = 3 + \frac{(-1)^n}{n}$ converge to in $\mathbb{R}_\ell$?

Solution: (a) $\mathbb{R}_\ell$ is the topological space with basis given by all half-open intervals in $\mathbb{R}$ of the form $[a, b)$.

(b) We claim that x_n does not converge to any point in $\mathbb{R}_\ell$. Indeed, let $x \in \mathbb{R}$. If $x < 3$, then for all even values of n, the point x_n is not contained in the neighborhood $[x, 3)$ of x, so that x_n does not converge to x. If $x \geq 3$, then for all odd values of n, the point x_n is not contained in the neighborhood $[x, x + 1)$ of x, so that x_n does not converge to x.

Comments: For (a), many students said that the topology on $\mathbb{R}_\ell$ was just the set of half-open intervals $[a, b)$; this is merely a basis for the lower limit topology. It's important to distinguish between a basis and the topology it generates.

For (b), there was a fair amount of confusion as to what it means to say that $x_n \to x$ for a sequence x_n in an arbitrary topological space. Unless you understand this notion precisely, problems like this are effectively impossible to solve. Some students also tried to argue that, if it exists, the limit of x_n would have to be 3 by saying that the points x_n are "getting closer and closer to 3". While such intuitive statements can be helpful, they strictly speaking don't make any sense ("closer and closer" is related to the idea of topology and, more specifically, having a metric). For example, if $\mathbb{R}$ has the finite complement topology, the same sequence x_n converges to any real number, despite still getting "closer and closer to 3".

Problem 2: [20 points] Let X be a set and let $\{\mathfrak{T}_\alpha\}_{\alpha \in I}$ be a nonempty family of topologies on X. Prove or disprove: the union $\bigcup_{\alpha \in I} \mathfrak{T}_\alpha$ is necessarily a topology on X.

Solution: This is false. Take $X = \{a, b, c\}$, $\mathfrak{T}_1 = \{X, \emptyset, \{a\}\}$, and $\mathfrak{T}_2 = \{X, \emptyset, \{b\}\}$. Then $\{a\}, \{b\} \in \mathfrak{T}_1 \cup \mathfrak{T}_2$ but $\{a\} \cup \{b\} \notin \mathfrak{T}_1 \cup \mathfrak{T}_2$, so $\mathfrak{T}_1 \cup \mathfrak{T}_2$ is not a topology on X.

Comments: This was a problem on the homework; there are many possible counterexamples. Some students tried to repeat the above argument with $X = \{a, b\}$; this does not work. (Do you see why not?) There was some notational sloppiness here.

Remember: a topology is a set of sets. In particular, things like a shouldn't be in your topologies; rather, they contain things like $\{a\}$.

Problem 3: [15 points] Endow the set $\mathbb{R} \times \mathbb{Z}$ with the dictionary order $<$. That is, for $(x, r), (x', r') \in \mathbb{R} \times \mathbb{Z}$, we have $(x, r) < (x', r')$ if $x < x'$ (as real numbers) or $x = x'$ and $r < r'$ (as integers). Prove that the order topology on $\mathbb{R} \times \mathbb{Z}$ is the discrete topology.

Solution: It suffices to show that every singleton subset of $\mathbb{R} \times \mathbb{Z}$ is open in the order topology. Indeed, if $(x, r) \in \mathbb{R} \times \mathbb{Z}$, we have that $\{(x, r)\}$ coincides with the open interval $((x, r-1), (x, r+1))$, and is therefore open in the order topology. Therefore, the order topology on $\mathbb{R} \times \mathbb{Z}$ is the discrete topology.

Comments: This problem required that you understand what the order topology on a set is. Some students seemed to think that $\mathbb{R} \times \mathbb{Z}$ has a largest (or a smallest) element; can you see why it does not?

Problem 4: [5 + 15 points] Let X be a topological space. (a) Carefully define what it means for X to be "Hausdorff". (b) Give an example of a topological space X which is not Hausdorff but which has the "T_1 property" that $\{x\}$ is a closed set for all $x \in X$.

Solution: (a) X is Hausdorff if for all $x, y \in X$ with $x \neq y$, there exist neighborhoods U of x and V of y such that $U \cap V = \emptyset$.

(b) Consider $\mathbb{R}$ with the finite complement topology. Then for any $x \in \mathbb{R}$ we have that $\mathbb{R} - (\mathbb{R} - \{x\}) = \{x\}$ is finite, so that $\{x\}$ is a closed set and $\mathbb{R}$ has the T_1 property. On the other hand, let U and V be neighborhoods of $0, 1 \in \mathbb{R}$, respectively. Then $\mathbb{R} - (U \cap V) = (\mathbb{R} - U) \cup (\mathbb{R} - V)$ is a finite set (since neither U nor V are empty). Since $\mathbb{R}$ is infinite, this means that there exists $x \in U \cap V$ and U and V are not disjoint. We conclude that $\mathbb{R}$ with the finite complement topology is not Hausdorff.

Comments: There were some students who didn't know the definition of a Hausdorff space, or who didn't quite get the definition right (the order and nature of the quantifiers is crucial here). Most students who got Part (b) correct gave the example of $\mathbb{R}$ with the finite complement topology. Some students tried to use any set X with the finite complement topology; this works if and only if X is infinite. If X is finite, the finite complement topology on X is the discrete topology, which is Hausdorff.

Problem 5: [5 + 15 points] Let X and Y be topological spaces and let $y_0 \in Y$ be a point. (a) Give a basis for the "product topology" on $X \times Y$. (b) Prove that the function $f : X \to X \times Y$ defined by $f(x) = (x, y_0)$ is continuous (where $X \times Y$ is endowed with the product topology).

Solution: (a) A basis for the product topology on $X \times Y$ is all sets of the form $U \times V$, where $U \subset X$ is open and $V \subset Y$ is open.

(b) Let $U \times V$ be a typical basis element of the product topology on $X \times Y$, so that $U \subset X$ and $V \subset Y$ are open. It suffices to show that $f^{-1}(U \times V)$ is open in X. Indeed, we have that $f^{-1}(U \times V) = \emptyset$ if $y_0 \notin V$ and $f^{-1}(U \times V) = U$ if $y_0 \in V$. In either case, $f^{-1}(U \times V)$ is open in X, as desired.

Comments: There was some sloppiness here in defining the product topology. In particular, some students forgot to specify that U and V must be open sets. There was also some confusion about the definition of continuity. Some students tried to define a function $f^{-1} : X \times Y \to X$; if Y has more than one point, the function f given here isn't invertible. The notation $f^{-1}(U \times V)$ does not mean that f is an invertible function. Some students also didn't realize that the set $f^{-1}(U \times V)$ could be empty if $y_0 \notin V$.

Problem 6: [15 points] Consider the real numbers $\mathbb{R}$ with their standard topology and endow the closed interval $[-1, 2] \subset \mathbb{R}$ with the subspace topology. Consider the following subsets of $[-1, 2]$: $A = (0, 1), B = (1, 2], C = \{-1\} \cup (1, 2]$. Which are open in $\mathbb{R}$? Which are open in $[-1, 2]$?

Solution: A is an open interval, and hence open in $\mathbb{R}$, and therefore also in the subspace $[-1, 2]$. B is not open in $\mathbb{R}$ because no open neighborhood of $2 \in B$ is contained in B. On the other hand, we have that $B = [-1, 2] \cap (1, 3)$, so that B is open in $[-1, 2]$. C is not open in $[-1, 2]$ because any neighborhood of -1 in $[-1, 2]$ would contain points slightly larger than -1, which would not be contained in C. C is therefore not open in $\mathbb{R}$.

Comments: This was an example from class. Students did well here in general.