Math 190: Fall 2014
Midterm 2
Solutions and Comments

Instructions: This is a 50 minute exam. No books, notes, electronic devices, or interpersonal assistance are allowed. Write your answers in your blue book, making it clear what problem you are working on. Be sure to justify your answers. This exam is out of 95 points; you get 5 points for legibly writing your name on your blue book. Good luck!

Problem 1: [10 points] Prove that $[0, 1)$ and S^1 are not homeomorphic.

Solution: We know that $[0, 1)$ is not compact (since $\{[0, \frac{n}{n+1}) : n = 1, 2, 3, \dots\}$ is an open cover of $[0, 1)$ which does not have a finite subcover) and we know that S^1 is compact (as a closed and bounded subspace of $\mathbb{R}^2$). Therefore, $[0, 1)$ and S^1 are not homeomorphic.

Alternatively, suppose that $f : [0, 1) \to S^1$ were a homeomorphism. Then the restriction $f|_{[0,1)-\{1/2\}} : [0, 1)-\{1/2\} \to S^1-\{f(1/2)\}$ would also be a homeomorphism. But $[0, 1) - \{1/2\}$ is not connected and $S^1 - \{f(1/2)\}$ is connected.

Comments: This was an example I went over in class; I gave the second "connectedness" argument presented here.

Most students attempted the connectedness argument presented above. There was a fair amount of confusion here; some students felt that $S^1 - \{f(1/2)\}$ (i.e., the circle minus a single point) was a disconnected space. You should convince yourself that it is homeomorphic to $(0, 1)$ (and hence connected). Other students did not consider the restriction, which is necessary for this argument to work (since both $[0, 1)$ and S^1 are connected). Other students said that the continuous bijection $[0, 1) \to S^1$ given by $t \mapsto e^{2\pi it}$ is not a homeomorphism. It isn't, but that in and of itself doesn't mean that $[0, 1)$ and S^1 are not homeomorphic. Moreover, since $[0, 1)$ isn't compact, we can't apply the result that a continuous bijection $X \to Y$ with X compact and Y Hausdorff is automatically a homeomorphism.

Problem 2: [5 + 10 points] (a) Carefully define what it means for a space X to be "compact". (b) Is $[0, 1]$ compact as a subspace of $\mathbb{R}_\ell$?

Solution: (a) X is compact if for every open cover $\mathcal{U}$ of X, there are finitely many sets $U_1, U_2, \dots, U_n \in \mathcal{U}$ such that $X = U_1 \cup U_2 \cup \cdots \cup U_n$.

(b) No. Since $[1, 2)$ is open in $\mathbb{R}_\ell$, we know that $\{1\}$ is open in $[0, 1]$ as a subspace of $\mathbb{R}_\ell$. We conclude that

$$\left\{ \left[0, \frac{n}{n+1}\right) : n = 1, 2, 3, \dots \right\} \cup \{\{1\}\}$$

is an open cover of $[0, 1]$ with no finite subcover.

Comments: As always, it is crucial to know your definitions perfectly. A large fraction of students did not give the correct definition of compactness. Some felt that X is compact if X possesses a finite open cover (then *any* space would be compact, as $\{X\}$ is an open cover of X). Other students omitted the term "open" in the definition, so X is compact if every cover of X has a finite subcover (then *no* infinite space X would be compact). Without knowing the definition of compactness, it is impossible to use it in examples (such as Part b). Compactness is one of the fundamental concepts in point-set topology (among other fields); if you do not understand it, please ask questions in class or come to office hours.

Problem 3: [10 points] Prove that an infinite set X is connected in the finite complement topology.

Solution: Suppose $X = U \cup V$ is a separation of X. Since U and V are nonempty, we know that $X - U$ and $X - V$ are finite. But $X - U = V$ and $X - V = U$, so $X = U \cup V = (X - V) \cup (X - U)$ is finite, which is a contradiction.

Comments: This was a problem from the homework. Students did quite well in general. There was some confusion as to the definition of the finite complement topology.

Problem 4: [10 points] Prove that every metric space is Hausdorff.

Solution: Let (X, d) be a metric space and let $x, y \in X$ with $x \neq y$. Then $d(x, y) > 0$. Let $\epsilon = \frac{d(x,y)}{2}$. Then $B_d(x, \epsilon)$ and $B_d(y, \epsilon)$ are open neighborhoods of x and y, respectively. If $z \in B_d(x, \epsilon) \cap B_d(y, \epsilon)$, the triangle inequality implies that

$$d(x, y) \leq d(x, z) + d(z, y) < \epsilon + \epsilon = d(x, y),$$

which is a contradiction. We conclude that $B_d(x, \epsilon)$ and $B_d(y, \epsilon)$ are disjoint.

Comments: This was a fact I proved in class. Of course, ϵ could be taken to be smaller than $\frac{d(x,y)}{2}$. Given the key use of the triangle inequality in this proof, the words "triangle inequality"

Problem 5: [15 points] Consider the rationals $\mathbb{Q}$ as a subspace of $\mathbb{R}$. Is the quotient space $\mathbb{R}/\mathbb{Q}$ Hausdorff? Justify your answer.

Solution: No. Let $x \in \mathbb{R} - \mathbb{Q}$ and let $[x] \in \mathbb{R}/\mathbb{Q}$ be the corresponding (singleton) equivalence class. Let U be a neighborhood of $[x]$ in $\mathbb{R}/\mathbb{Q}$. We claim $\mathbb{Q} \in U$. Indeed, we know that $\tilde{U} := \{y \in \mathbb{R} : [y] \in U\}$ is open in $\mathbb{R}$ and contains x. Therefore, $\tilde{U}$ contains some small open interval $(x - \epsilon, x + \epsilon)$. Since there exists a rational number in this interval, we conclude that $\tilde{U} \cap \mathbb{Q} \neq \emptyset$, so that $\mathbb{Q} \in U$. Therefore, $\mathbb{Q}$ is in *every* nonempty open set in $\mathbb{R}/\mathbb{Q}$, so that $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.

Comments: This was an example from class. Most students said that $\mathbb{R}/\mathbb{Q}$ isn't Hausdorff. There were substantial issues with notation. *Points* in $\mathbb{R}/\mathbb{Q}$ are *sets of points* in $\mathbb{R}$. So it's incorrect to say that (for example) $\sqrt{2} \in \mathbb{R}/\mathbb{Q}$. On the other hand, we have that $[\sqrt{2}] = \{\sqrt{2}\} \in \mathbb{R}/\mathbb{Q}$. We also have that $\mathbb{Q} \in \mathbb{R}/\mathbb{Q}$. Be sure that you understand the notation in this proof.

Problem 6: [5 + 25 points] Define an equivalence relation $\sim$ on $\mathbb{R}^2$ by

$$(x_1, y_1) \sim (x_2, y_2) \text{if and only if } x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

(a) Describe the *points* of the quotient space $\mathbb{R}^2/\sim$. (Hint: Points in $\mathbb{R}^2/\sim$ are equivalence classes in $\mathbb{R}^2$.)

(b) Prove that the quotient space $\mathbb{R}^2/\sim$ is homeomorphic to the subspace $\mathbb{R}_{\geq 0}$ of $\mathbb{R}$.

Solution: (a) The points of $\mathbb{R}^2/\sim$ are the circles $\{x^2 + y^2 = r : r \geq 0\}$ centered at the origin (including the "circle of zero radius" $\{(0, 0)\}$ given by the origin itself).

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ by $f(x, y) = x^2 + y^2$. Then f is continuous and if $(x_1, y_1) \sim (x_2, y_2)$ we have $f(x_1, y_1) = f(x_2, y_2)$. By the Universal Property of the Quotient Topology, the function $\overline{f} : \mathbb{R}^2/\sim \to \mathbb{R}_{\geq 0}$ defined by $\overline{f} : [(x, y)] \mapsto x^2 + y^2$ is continuous.

Define $\overline{g} : \mathbb{R}_{\geq 0} \to \mathbb{R}^2/\sim$ by $\overline{g}(r) = [(\sqrt{r}, 0)]$. Then $\overline{g}$ is also continuous. We check that $\overline{f}$ and $\overline{g}$ are mutually inverse. Indeed, for any $r \in \mathbb{R}_{\geq 0}$ we have

$$\begin{aligned} \overline{f}(\overline{g}(r)) &= \overline{f}([(\sqrt{r}, 0)]) \\ &= (\sqrt{r})^2 + 0^2 \\ &= r. \end{aligned}$$

On the other hand, for any $(x, y) \in \mathbb{R}^2$ we have

$$\begin{aligned} \overline{g}(\overline{f}([(x, y)])) &= \overline{g}(x^2 + y^2) \\ &= [(\sqrt{x^2 + y^2}, 0)] \\ &= [(x, y)], \end{aligned}$$

where we used the fact that $(\sqrt{x^2 + y^2}, 0) \sim (x, y)$ in the last line. We conclude that $\overline{g}$ and $\overline{f}$ are mutually inverse, so that $\overline{f}$ is a homeomorphism and $\mathbb{R}^2/\sim$ and $\mathbb{R}_{\geq 0}$ are homeomorphic.

Comments: I was lenient about whether the origin itself counts as a "circle around the origin", or if the phrase "around the origin" was included in your solution to Part a – Part a was really there to help you with Part b.

Mastering the quotient topology is one of the key goals of any intro point set topology course. The quotient topology is used ad nauseum in algebraic topology (the key type of spaces considered there are formed by attaching various "cells" – copies of D^n – together) and also comes up a fair amount in differential topology/Lie theory. The lack of emphasis on the quotient topology is, in my opinion, the largest deficiency in your textbook.

The proof presented for showing that a "weird" quotient space (here, $\mathbb{R}^2/\sim$) is homeomorphic to a "well understood" space (here $\mathbb{R}_{\geq 0}$) is a standard argument. To show that $X/\sim$ is homeomorphic to Y, first build a continuous function $f : X \to Y$ (which will typically be "continuous from calculus" if you're using the Euclidean topology). You then check that f is constant on equivalence classes of X. This means that $\overline{f} : [x] \mapsto f(x)$ is a well-defined map *of sets* (this has nothing whatsoever to do with topology). Topology comes in when one uses the Universal Property of the Quotient Topology to say "since f is continuous, so is $\overline{f}$". Then one builds a second function $\overline{g} : Y \to X/\sim$ (the continuity of $\overline{g}$ is less mysterious – it is typically some continuous function $Y \to X$ composed with the canonical quotient map $X \to X/\sim$). To check that $\overline{g}$ and $\overline{f}$ are mutually inverse, one computes the compositions in both directions and checks that one gets the identity maps either way.

Many students had difficulty here. Most students managed to get a function $\overline{f} : \mathbb{R}^2/\sim \to \mathbb{R}_{\geq 0}$ defined by the formula above. Some students tried to show directly that $\overline{f}$ is injective and surjective. *Don't bother doing this.* In topology, continuous bijections need not have continuous inverses. So you need to compute a formula for the inverse anyways to show that it is continuous. Given formulas for $\overline{f}$ and its alleged inverse, checking that the alleged inverse works is completely routine; just compute compositions in either direction. This *automatically* gives you the fact that $\overline{f}$ is a bijection of sets. There was a great deal of confusion on this point, and some students who got all the points on this problem made their lives far too difficult.