Math 190: Winter 2016
Homework 3 Solutions
Due 5:00pm on Wednesday 1/27/2016

**Problem 1:** Let $X$ and $Y$ be topological spaces, let $A \subset X$ be a closed set in $X$, and let $B \subset Y$ be a closed set in $Y$. Prove that $A \times B$ is a closed subset of $X \times Y$ (in the product topology).

**Solution:** We know that $X - A$ is open in $X$ and $Y - B$ is open in $Y$. We can express $(X \times Y) - (A \times B)$ as a union

$$(X \times Y) - (A \times B) = [(X - A) \times Y] \cup [X \times (Y - B)]$$

of two basic open sets in the product $X \times Y$. Since $(X \times Y) - (A \times B)$ is open in $X \times Y$, it follows that $A \times B$ is closed in $X \times Y$.

**Problem 2:** Let $X$ be a topological space and let $A, B$, and $A_\alpha$ be subsets of $X$.

1. If $A \subset B$, prove that $\overline{A} \subset \overline{B}$.
2. We have $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
3. We have $\bigcup_{\alpha} \overline{A_\alpha} \subset \overline{\bigcup_{\alpha} A_\alpha}$. Show that equality does not necessarily hold here.

**Solution:**

1. We have that

$$\overline{A} = \bigcap_{A \subset C, C \text{ closed}} C \subset \bigcap_{B \subset C, C \text{ closed}} C = \overline{B}.$$ 

2. By Part (1), we have that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. If $x \in X$ does not lie in $\overline{A \cup B}$, then there exist neighborhoods $U$ and $V$ of $x$ such that $U \cap A = \emptyset$ and $V \cap B = \emptyset$. But then $U \cap V$ is a neighborhood of $x$ with $(U \cap V) \cap (A \cup B) = \emptyset$. This implies that $x \notin \overline{A \cup B}$.

3. Since $A_{\alpha_0} \subset \bigcup_{\alpha} A_\alpha$ for all $\alpha_0$, the claimed containment follows from Part (1). To see that equality need not hold, give $\mathbb{R}$ the standard topology and consider the sets $\{\{r\} : r \in \mathbb{Q}\}$. Then $\{\{r\} = \{r\}$ for every $r \in \mathbb{Q}$, so that $\bigcup_{r \in \mathbb{Q}} \{r\} = \mathbb{Q}$. On the other hand, we have that $\bigcup_{r \in \mathbb{R}} \{r\} = \mathbb{Q} = \mathbb{R}$.

**Problem 3:** Let $A \subset X$ and $B \subset Y$, where $X$ and $Y$ are topological spaces. Prove that $\overline{A \times B} = \overline{A} \times \overline{B}$ inside $X \times Y$ (where $X \times Y$ has the product topology).

**Solution:** Suppose $(x, y) \in \overline{A \times B}$. Let $W$ be a neighborhood of $(x, y)$ in $X \times Y$. Then there exists a neighborhood $U$ of $x$ in $X$ and a neighborhood $V$ of $y$ in $Y$ such that $(x, y) \in U \times V \subset W$. Since $x \in \overline{A}$, it follows that $U \cap A \neq \emptyset$. Since $y \in \overline{B}$, it follows that $V \cap B \neq \emptyset$. We conclude that $(U \times V) \cap (A \times B) \neq \emptyset$, so that $W \cap (A \times B) \neq \emptyset$. We conclude that $(x, y) \in \overline{A \times B}$.

Suppose $(x, y) \in \overline{A \times B}$. Let $U$ be a neighborhood of $A$ in $x$. Then $U \times Y$ is a neighborhood of $(x, y) \in X \times Y$, so that $U \times Y$ meets $A \times B$. But this forces $U$ to meet $A$, so that $x \in \overline{A}$. Similarly, $y \in \overline{B}$, so that $(x, y) \in \overline{A \times B}$.

**Problem 4:** Prove that a product of two Hausdorff topological spaces is Hausdorff.
Solution: Let $X$ and $Y$ be Hausdorff spaces and let $(x_1, y_1)$ and $(x_2, y_2)$ be two distinct points in $X \times Y$. These points must disagree in at least one of their coordinates; without loss of generality assume $x_1 \neq x_2$. There exist neighborhoods $U_1$ of $x_1$ and $U_2$ of $x_2$ in $X$ such that $U_1 \cap U_2 = \emptyset$. Now $U_1 \times Y$ and $U_2 \times Y$ are neighborhoods of $(x_1, y_1)$ and $(x_2, y_2)$ in $X \times Y$ (respectively), and

$$(U_1 \times Y) \cap (U_2 \times Y) = (U_1 \cap U_2) \times Y = \emptyset \times Y = \emptyset.$$  

We conclude that $X \times Y$ is Hausdorff.

Problem 5: Let $X$ be a topological space and suppose $A \subset Y \subset X$. Give $Y$ the subspace topology. Prove that the topology that $A$ inherits as a subspace of $X$ equals the topology that $A$ inherits as a subspace of $Y$.

Solution: Let $\mathcal{T}$ be the topology on $A$ as a subspace of $X$ and let $\mathcal{T}'$ be the topology on $A$ as a subspace of $Y$.

If $U \in \mathcal{T}$, then there exists $V \subset X$ open in $X$ such that $U = V \cap A$. But then $V \cap Y$ is open in $Y$, so that $U = V \cap A = (V \cap Y) \cap A \in \mathcal{T}'$.

If $U \in \mathcal{T}'$, there exists $W \subset Y$ open in $Y$ such that $U = W \cap A$. But then there exists $V \subset X$ open in $X$ such that $W = V \cap Y$. We have that $U = W \cap A = V \cap (Y \cap A) = V \cap A \in \mathcal{T}$.

Problem 6: Let $X$ be a topological space and give $X \times X$ the product topology. The diagonal $\Delta \subset X \times X$ is

$$\Delta := \{(x, x) : x \in X\}.$$  

Prove that $X$ is Hausdorff if and only if $\Delta$ is closed in $X \times X$.

Solution: Suppose $X$ is Hausdorff. Let $(x, y) \in X \times X - \Delta$, so that $x \neq y$. Let $U$ be a neighborhood of $x$ and $V$ be a neighborhood of $y$ so that $U \cap V = \emptyset$. Then $U \times V$ is a neighborhood of $(x, y)$ in $X \times X$ and $(U \times V) \cap \Delta = \emptyset$. This means $(x, y) \notin \Delta$, forcing $\overline{\Delta} = \Delta$, so that $\Delta$ is closed in $X \times X$.

Suppose $\Delta$ is closed in $X \times X$. Let $x, y \in X$ with $x \neq y$. Then $(x, y) \notin \Delta$. This means we can find a neighborhood $W$ of $(x, y)$ in $X \times X$ such that $W \cap \Delta = \emptyset$. By the definition of the product topology, we can find a neighborhood $U$ of $x$ and a neighborhood $V$ of $y$ such that $(x, y) \in (U \times V) \subset W$. This forces $(U \times V) \cap \Delta = \emptyset$, so that $U \cap V = \emptyset$. We conclude that $X$ is Hausdorff.

Problem 7: In the finite complement topology on $\mathbb{R}$, to what point or points (if any) does the sequence $x_n = 3 + \frac{(-1)^n}{n}$ converge? What about in the lower limit topology $\mathbb{R}_\ell$?

Solution: We claim that in the finite complement topology, we have that $x_n \to x$ for any $x \in \mathbb{R}$. To see this, let $x \in \mathbb{R}$ and let $U$ be a neighborhood of $x$ in the finite complement topology. Then $\mathbb{R} - U$ is a finite set. In particular, there exists $N$ such that $n > N$ implies $x_n \notin \mathbb{R} - U$, forcing $x_n \in U$. We conclude that $x_n \to x$.

We claim that in $\mathbb{R}_\ell$, the sequence $x_n$ does not converge to any point. To see this, let $x \in \mathbb{R}$. If $x \neq 3$, we can choose an open interval $(a, b)$ such that $x \in (a, b)$ but $3 \notin (a, b)$. If $x < 3$, the interval $(a, b)$ does not contain $x_n$ for any even value of $n$. If $x > 3$, the interval $(a, b)$ does not contain $x_n$ for any odd value of $n$. Since $(a, b)$ is open in $\mathbb{R}_\ell$, we conclude that $x_n$ does not converge to $x$. To see that $x_n$ does not converge
to 3, consider the neighborhood \([3, 4]\) of 3 in \(\mathbb{R}_ℓ\). For \(n\) odd, we have \(x_n \notin [3, 4]\). It follows that \(x_n\) does not converge to 3 in \(\mathbb{R}_ℓ\).

**Problem 8:** Let \(f : \mathbb{R}^m \to \mathbb{R}^n\) be a function (where the spaces have the standard topology). In analysis, you learned the following definition of \(f\) being ‘continuous’:

“For every \(x \in \mathbb{R}^m\) and every \(\epsilon > 0\), there exists \(\delta > 0\) such that for all \(y \in \mathbb{R}^m\) with \(|x - y| < \delta\), we have that \(|f(x) - f(y)| < \epsilon\).”

In topology, we have the following definition of \(f\) being ‘continuous’:

“For every open set \(U \subset \mathbb{R}^n\), we have that \(f^{-1}(U)\) is open in \(\mathbb{R}^m\).”

Prove that these two definitions are equivalent.

**Solution:** Suppose that \(f\) is continuous with respect to the \(\epsilon - \delta\) definition. Let \(U \subset \mathbb{R}^n\) be an open set and let \(x \in f^{-1}(U)\). There exists \(\epsilon > 0\) such that \(B(f(x), \epsilon) \subset U\). Choose \(\delta > 0\) such that \(|x - y| < \delta\) implies \(|f(x) - f(y)| < \epsilon\). Said differently, we have that \(y \in B(x, \delta)\) implies \(f(y) \in B(f(x), \epsilon) \subset U\). This forces \(B(x, \delta) \subset f^{-1}(U)\). We conclude that \(f^{-1}(U)\) is open in \(\mathbb{R}^m\).

Suppose that \(f\) is continuous with respect to the open sets definition. Let \(x \in \mathbb{R}^m\) and let \(\epsilon > 0\). Then \(B(f(x), \epsilon) \subset \mathbb{R}^n\) is an open set, so that \(f^{-1}(B(f(x), \epsilon))\) is open in \(\mathbb{R}^m\). This means there exists \(\delta > 0\) so that \(B(x, \delta) \subset f^{-1}(B(f(x), \epsilon))\). But then \(f(B(x, \delta)) \subset B(f(x), \epsilon)\). Said differently, we have that \(|x - y| < \delta\) implies \(|f(x) - f(y)| < \epsilon\).

**Problem 9:** Let \(X\) and \(Y\) be topological spaces and let \(x_0 \in X\). Prove that the map

\[ f : Y \to X \times Y \]

given by \(f(y) = (x_0, y)\) is an imbedding.

**Solution:** Observe that the image of \(f\) is equal to \(\{x_0\} \times Y \subset X \times Y\). Let \(\pi : X \times Y \to Y\) be the canonical projection \(\pi(x, y) = y\). Then \(\pi\) is continuous and therefore the restriction \(\pi|_{\{x_0\} \times Y}\) is also continuous. For any \(y\) in \(Y\) we have

\[ \pi|_{\{x_0\} \times Y} \circ f(y) = \pi|_{\{x_0\} \times Y}(x_0, y) = y \]

and

\[ f \circ \pi|_{\{x_0\} \times Y}(x_0, y) = f(y) = (x_0, y). \]

It remains to check that \(f\) is continuous. Indeed, let \(U \times V \subset X \times Y\) be a basic open set (where \(U \subset X\) and \(V \subset Y\) are open). We have that

\[ f^{-1}(U \times V) = \begin{cases} V & x_0 \in U \\ \emptyset & x_0 \notin U. \end{cases} \]

In either case, we conclude that \(f^{-1}(U \times V)\) is open in \(Y\). Therefore, the function \(f\) is continuous and is a homeomorphism onto its image.

**Problem 10:** (Optional) This problem will introduce you to the Zariski topology on \(\mathbb{R}^n\). This is probably the most important non-Hausdorff topology on \(\mathbb{R}^n\).
A polynomial $f(x_1, \ldots, x_n)$ in $n$ variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{R}$ is a finite expression of the form

$$f(x_1, \ldots, x_n) = \sum_{(a_1, \ldots, a_n)} c_{a_1, \ldots, a_n} x_1^{a_1} \cdots x_n^{a_n},$$

where the $a_i$ are nonnegative integers and the $c_{a_1, \ldots, a_n}$ belong to $\mathbb{R}$. Let $\mathbb{R}[x_1, \ldots, x_n]$ denote the set of all polynomials in the variables $x_1, \ldots, x_n$ with real coefficients.

If $f(x_1, \ldots, x_n)$ is a polynomial with real coefficients, a zero of $f$ is a point $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that $f(\alpha_1, \ldots, \alpha_n) = 0$. We let $V(f) \subset \mathbb{R}^n$ denote the set of all zeros of $f$.

That is,

$$V(f) = \{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n : f(\alpha_1, \ldots, \alpha_n) = 0 \}.$$

($V$ is for ‘vanishing’ or ‘variety’.) More generally, if $E \subset \mathbb{R}[x_1, \ldots, x_n]$ is any (nonempty) set of polynomials, let $V(E) = \bigcap_{f \in E} V(f)$ be the set of common zeroes of these polynomials.

1. In $\mathbb{R}^2$, sketch $V(x_1^2 + 4x_2^2 - 16)$ and $V(x_1 + x_2 - 1, x_1 - 4)$.

2. Prove that the collection of subsets

$$\{ V(E) : E \subset \mathbb{R}[x_1, \ldots, x_n] \}$$

satisfy the axioms for the closed sets of a topology on $\mathbb{R}^n$. This is called the Zariski topology on $\mathbb{R}^n$.

3. Prove that the Zariski topology on $\mathbb{R}^1$ is the finite complement topology. (Hint: A nonconstant polynomial in $\mathbb{R}[x]$ only has finitely many roots.)

4. Prove that the Zariski topology on $\mathbb{R}^n$ for $n > 1$ is not the finite complement topology.

5. Prove that the Zariski topology on $\mathbb{R}^2$ is not the product $\mathbb{R}^1 \times \mathbb{R}^1$ of the Zariski topology on $\mathbb{R}^1$ with itself.

6. Prove that the Zariski topology is not Hausdorff.

7. Prove that the Euclidean topology is finer than the Zariski topology.

For those who have taken algebra, the Zariski topology may also be defined on $k^n$, where $k$ is any field. The Zariski topology is most interesting when $k$ is algebraically closed.