Problem 1: Let \( \text{Mat}(n, \mathbb{R}) \) denote the set of \( n \times n \) real matrices, which we can identify with the Euclidean space \( \mathbb{R}^{n^2} \). Explain why the determinant function 
\[
\det : \text{Mat}(n, \mathbb{R}) \to \mathbb{R}
\]
is continuous.

Problem 2: Let \( GL_n(\mathbb{R}) \subset \text{Mat}(n, \mathbb{R}) \) be the subset of invertible \( n \times n \) matrices. Prove that \( GL_n(\mathbb{R}) \) is open. (So, if a matrix \( A \) is invertible, there exists \( \epsilon > 0 \) such that altering any entry of \( A \) by at most \( \epsilon \) results in an invertible matrix.)

Problem 3: A subset \( D \) of a space \( X \) is called dense if \( \overline{D} = X \). Let \( X \) and \( Y \) be spaces with \( Y \) Hausdorff. Let \( D \subset X \) be a dense subset. Let 
\[
f, g : X \to Y
\]
be continuous functions such that \( f(x) = g(x) \) for all \( x \in D \). Prove that \( f = g \).

Problem 4: Prove that the conclusion of the last problem need not hold if \( Y \) is not Hausdorff.

Problem 5: Look up “stereographic projection”. Let \( S^n \) be the \( n \)-sphere and let
\[
N = (1,0,\ldots,0)
\]
be the ‘north pole’. Prove that \( \mathbb{R}^n \) is homeomorphic to \( S^n - \{N\} \).

Problem 6: Let \( A \subset \mathbb{R}^2 \) be the annulus
\[
A = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}.
\]
Prove that \( A \) is homeomorphic to \( S^1 \times I \), where \( I = [0,1] \) is the unit interval.

Problem 7: Let \( f : A \to B \) and \( g : C \to D \) be continuous functions. Define a new function 
\[
f \times g : A \times C \to B \times D
\]
by
\[
f \times g : a \times c \mapsto f(a) \times g(c).
\]
Prove that \( f \times g \) is a continuous function.

Problem 8: Let \( f : X \to Y \) be a function between topological spaces. If \( X \) is discrete, prove that \( f \) is continuous. If \( Y \) is indiscrete, prove that \( f \) is continuous.

Problem 9: Let \( f : X \to Y \) be a continuous function. The graph of \( f \) is the subspace \( \Gamma_f \subset X \times Y \) given by 
\[
\Gamma_f := \{(x,f(x)) : x \in X\}.
\]
Prove that \( X \) is homeomorphic to \( \Gamma_f \).
Problem 10: (Optional) This problem is about continuous maps in the context of the Zariski topology introduced on the optional problem of the last homework.

A Zariski closed subset $V \subseteq \mathbb{R}^n$ is called a variety. Recall that this means that $V$ is the set of common zeros $V(E)$ of some collection $E$ of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$.

1) Give some examples of varieties that you have seen before in high school. Prove that the set of singular $n \times n$ real matrices is a variety.

Suppose $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ are varieties. A polynomial map $\psi : V \rightarrow W$ is a function given by a list $\psi = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ of $m$ polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ with the property that, for all $(a_1, \ldots, a_n) \in V$ we have $\psi(a_1, \ldots, a_n) = (f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n)) \in W$.

2) Suppose that $V = S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 1 = 0\}$ is the unit circle. Give two different polynomials $f(x, y), g(x, y) \in \mathbb{R}[x, y]$ such that the polynomial maps $S^1 \rightarrow \mathbb{R}$ induced by $f$ and $g$ coincide.

3) Suppose $\psi : V \rightarrow W$ is a polynomial map between varieties $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$. Prove that $\psi$ is continuous in the Zariski topology (i.e., the subspace topologies on $V$ and $W$ inherited from the Zariski topologies on $\mathbb{R}^n$ and $\mathbb{R}^m$).