Math 190: Winter 2016
Homework 4
Due 5:00pm on Wednesday 2/3/2016

Problem 1: Let $\mathrm{Mat}(n, \mathbb{R})$ denote the set of $n \times n$ real matrices, which we can identify with the Euclidean space $\mathbb{R}^{n^2}$. Explain why the determinant function

$$\det : \mathrm{Mat}(n, \mathbb{R}) \to \mathbb{R}$$

is continuous.

Problem 2: Let $GL_n(\mathbb{R}) \subset \mathrm{Mat}(n, \mathbb{R})$ be the subset of *invertible* $n \times n$ matrices. Prove that $GL_n(\mathbb{R})$ is open. (So, if a matrix A is invertible, there exists $\epsilon > 0$ such that altering any entry of A by at most ϵ results in an invertible matrix.)

Problem 3: A subset D of a space X is called *dense* if $\overline{D} = X$. Let X and Y be spaces with Y Hausdorff. Let $D \subset X$ be a dense subset. Let

$$f, g : X \to Y$$

be continuous functions such that $f(x) = g(x)$ for all $x \in D$. Prove that $f = g$.

Problem 4: Prove that the conclusion of the last problem need not hold if Y is not Hausdorff.

Problem 5: Look up "stereographic projection". Let S^n be the n-sphere and let

$$N = (1, 0, \ldots, 0)$$

be the 'north pole'. Prove that $\mathbb{R}^n$ is homeomorphic to $S^n - \{N\}$.

Problem 6: Let $A \subset \mathbb{R}^2$ be the annulus

$$A = \{(x, y) \in \mathbb{R}^2 \,:\, 1 \leq x^2 + y^2 \leq 4\}.$$

Prove that A is homeomorphic to $S^1 \times I$, where $I = [0, 1]$ is the unit interval.

Problem 7: Let $f : A \to B$ and $g : C \to D$ be continuous functions. Define a new function

$$f \times g : A \times C \to B \times D$$

by

$$f \times g : a \times c \mapsto f(a) \times g(c).$$

Prove that $f \times g$ is a continuous function.

Problem 8: Let $f : X \to Y$ be a function between topological spaces. If X is discrete, prove that f is continuous. If Y is indiscrete, prove that f is continuous.

Problem 9: Let $f : X \to Y$ be a continuous function. The *graph* of f is the subspace $\Gamma_f \subset X \times Y$ given by

$$\Gamma_f := \{(x, f(x)) \,:\, x \in X\}.$$

Prove that X is homeomorphic to Γ_f.

Problem 10: (Optional) This problem is about continuous maps in the context of the Zariski topology introduced on the optional problem of the last homework.

A Zariski closed subset $V \subseteq \mathbb{R}^n$ is called a *variety*. Recall that this means that V is the set of common zeros $\mathbf{V}(E)$ of some collection E of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$.

(1) Give some examples of varieties that you have seen before in high school. Prove that the set of singular $n \times n$ real matrices is a variety.

Suppose $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ are varieties. A *polynomial map* $\psi : V \to W$ is a function given by a list $\psi = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ of m polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ with the property that, for all $(a_1, \ldots, a_n) \in V$ we have $\psi(a_1, \ldots, a_n) = (f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n)) \in W$.

(2) Suppose that $V = S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 1 = 0\}$ is the unit circle. Give two different polynomials $f(x, y), g(x, y) \in \mathbb{R}[x, y]$ such that the polynomial maps $S^1 \to \mathbb{R}$ induced by f and g coincide.

(3) Suppose $\psi : V \to W$ is a polynomial map between varieties $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$. Prove that ψ is continuous in the Zariski topology (i.e., the subspace topologies on V and W inherited from the Zariski topologies on $\mathbb{R}^n$ and $\mathbb{R}^m$).